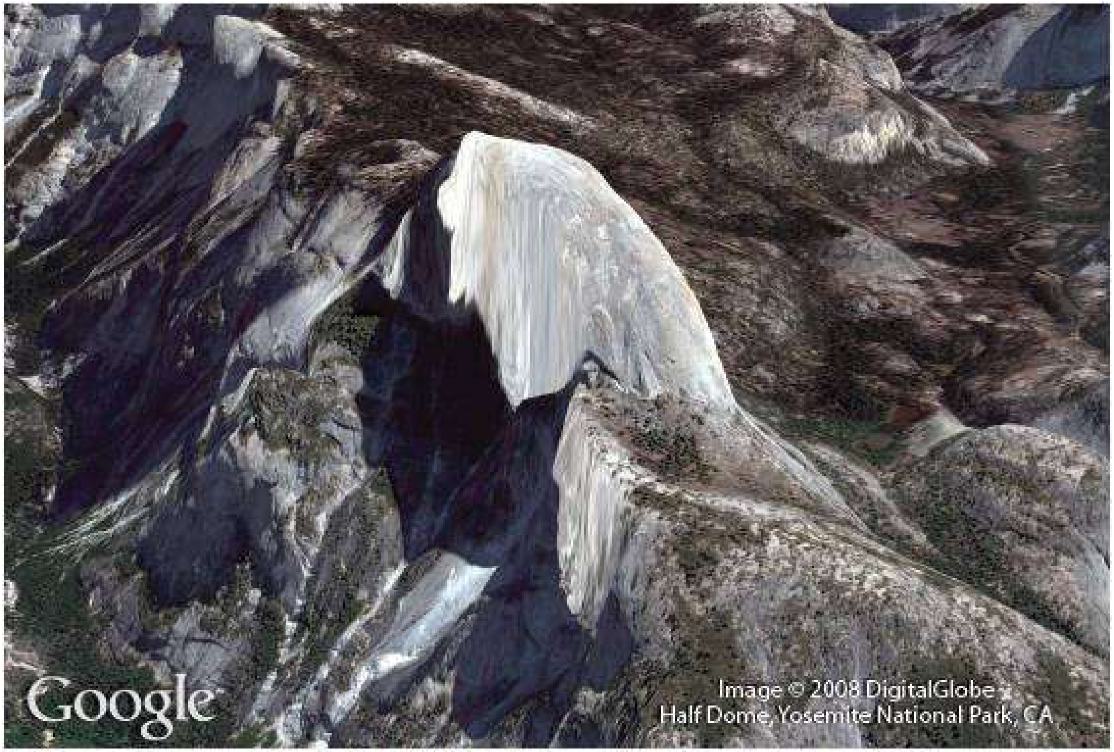
Exponents and Polynomials

9



SECTION 9.4

Negative Exponents

Objectives

A Simplify expressions containing negative exponents.

B Understand and apply the division property for exponents.

C Zero and one as exponents.

Negative Exponents

The time it takes light to travel one meter is approximately 3×10^{-9} seconds.

This speed is given in an expression that uses scientific notation with a negative exponent.

Scientific notation is a way we write very large or, in this case, very small numbers in a more manageable form.

A Negative Exponents

Negative Exponents

Definition

For **negative exponents,** if r is a positive integer and a is any number other than zero, then

$$a^{-r} = \frac{1}{a^r} \qquad a \neq 0$$

When we apply the definition for negative exponents to expressions containing negative exponents, we end up with reciprocals.

Negative Exponents

To check your understanding of negative exponents, look over the two lines below.

$$2^1 = 2 \qquad 2^2 = 4 \qquad 2^3 = 8 \qquad 2^4 = 16$$

$$2^{-1} = \frac{1}{2} \qquad 2^{-2} = \frac{1}{4} \qquad 2^{-3} = \frac{1}{8} \qquad 2^{-4} = \frac{1}{16}$$

The properties of exponents hold for negative exponents as well as positive exponents.

Negative Exponents

For example, the multiplication property for exponents, is still written as

$$a^r \cdot a^s = a^{r+s}$$

but now *r* and *s* can be negative numbers also.

Example 4

Simplify: $2^5 \cdot 2^{-7}$.

Solution:

This is multiplication with the same base, so we add exponents.

$2^5 \cdot 2^{-7} = 2^{5+(-7)}$	Multiplication property for exponents
$= 2^{-2}$	Add.
$= \dfrac{1}{2^2}$	Definition of negative exponents
$= \dfrac{1}{4}$	The square of 2 is 4.

B Division with Exponents

Division with Exponents

To develop our next property of exponents, we use the definition for positive exponents.

Consider the expression $\frac{X^6}{X^4}$.

We can simplify by expanding the numerator and denominator and then reducing to lowest terms by dividing out common factors.

$$\frac{X^6}{X^4} = \frac{X \cdot X \cdot X \cdot X \cdot X \cdot X}{X \cdot X \cdot X \cdot X} \qquad \text{Expand numerator and denominator.}$$

Division with Exponents

$$= \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}}$$

$$= x \cdot x$$

Divide out common factors.

$$= x^2$$

Write answer with exponent 2.

Note that the exponent in the answer is the difference of the exponents in the original problem.

More specifically, if we subtract the exponent in the denominator from the exponent in the numerator, we obtain the exponent in the answer.

Division with Exponents

This discussion leads us to another property of exponents.

Property **Division Property for Exponents**

If a is any number other than zero, and r and s are integers, then

$$\frac{a^r}{a^s} = a^{r-s}$$

In words: To divide two numbers with the same base, subtract the exponent in the denominator from the exponent in the numerator, and use the common base as the base in the answer.

Example 7

Divide: $10^{-3} \div 10^{5}$.

Solution:

We begin by writing the division problem in fractional form. Then we apply our division property.

$$10^{-3} \div 10^{5} = \frac{10^{-3}}{10^{5}} \quad \text{Write problem in fractional form.}$$

$$= 10^{-3-5} \quad \text{Division property for exponents}$$

$$= 10^{-8} \quad \text{Subtract.}$$

Example 7 – *Solution*

cont'd

$$= \frac{1}{10^8}$$ Definition of negative exponents

We can leave the answer in exponential form, as it is, or we can expand the denominator to obtain

$$= \frac{1}{100{,}000{,}000}$$

Division with Exponents

Expanded Distributive Property for Exponents

If a and b are any two real numbers ($b \neq 0$) and r is an integer, then

$$\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$$

In words: A quotient raised to a power is the quotient of the powers.

Example 9

Simplify the following expressions.

a. $\left(\frac{x}{2}\right)^3 = \frac{x^3}{2^3} = \frac{x^3}{8}$

b. $\left(\frac{5}{y}\right)^2 = \frac{5^2}{y^2} = \frac{25}{y^2}$

c. $\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4} = \frac{16}{81}$

C Zero and One as Exponents

Zero and One as Exponents

We have two special exponents left to deal with before our rules for exponents are complete: 0 and 1.

To obtain an expression for x^1, we will solve a problem two different ways.

$$\left.\begin{aligned} \frac{x^3}{x^2} &= \frac{x \cdot x \cdot x}{x \cdot x} = x \\ \frac{x^3}{x^2} &= x^{3-2} = x^1 \end{aligned}\right\} \quad \text{Hence } x^1 = x$$

Zero and One as Exponents

Stated generally, this rule says that $a^1 = a$.

This seems reasonable and we will use it since it is consistent with our property of division using the same base.

We use the same procedure to obtain an expression for x^0.

$$\left.\begin{aligned} \frac{5^2}{5^2} &= \frac{25}{25} = 1 \\ \frac{5^2}{5^2} &= 5^{2-2} = 5^0 \end{aligned}\right\} \quad \text{Hence } 5^0 = 1$$

Zero and One as Exponents

It seems, therefore, that the best definition of x^0 is 1 for all bases equal to x except $x = 0$. In the case of $x = 0$, we have 0^0, which we will not define.

This definition will probably seem awkward at first. Most people would like to define x^0 as 0 when they first encounter it.

Remember, the zero in this expression is an exponent, so x^0 does not mean to multiply by zero. Thus, we can make the general statement that $a^0 = 1$ for all real numbers except $a = 0$.

Zero and One as Exponents

Summarizing these results, we have our last properties for exponents.

Identity Property for Exponents

If a is any real number, then

$$a^1 = a$$

Zero Exponent Property

If a is any real number, then

$$a^0 = 1 \qquad a \neq 0$$

Example 10

Simplify the following expressions.

a. $8^0 = 1$

b. $8^1 = 8$

c. $4^0 + 4^1 = 1 + 4 = 5$

d. $(2x^2y)^0 = 1$

Example 11

Simplify: $3x^{-4} \cdot 5x$.

Solution:

To begin to simplify this expression, we regroup using the commutative and associative properties.

That way, the numbers 3 and 5 are grouped together, as are the powers of *x*. Note also how we write *x* as x^1, so we can see the exponent.

$$3x^{-4} \cdot 5x = 3x^{-4} \cdot 5x^1 \qquad \text{Write } x \text{ as } x^1.$$

Example 11 – *Solution*

cont'd

$= (3 \cdot 5)(x^{-4} \cdot x^{1})$ Commutative and associative properties

$= 15x^{-4+1}$ Multiply 3 and 5, then add exponents.

$= 15x^{-3}$ The sum of -4 and 1 is -3.

$= \dfrac{15}{x^{3}}$ Definition of negative exponents