Fractions and Mixed Numbers

3

SECTION 3.1

The Meaning and Properties of Fractions

Objectives

A Identify the numerator and denominator of a fraction.

B Identify proper and improper fractions.

C Write equivalent fractions.

D Simplify fractions with division.

E Compare the size of fractions.

The Meaning and Properties of Fractions

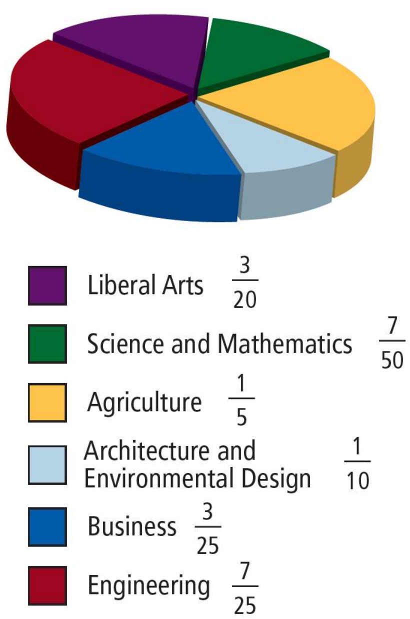
The information in the table below was taken from the website for California Polytechnic State University in San Luis Obispo, CA.

The pie chart was created from the table.

CAL POLY ENROLLMENT FOR FALL 2009	
School	**Fraction Of Students**
Agriculture	$\frac{1}{5}$
Architecture and Environmental Design	$\frac{1}{10}$
Business	$\frac{3}{25}$
Engineering	$\frac{7}{25}$
Liberal Arts	$\frac{3}{20}$
Science and Mathematics	$\frac{7}{50}$

Source: California Polytechnic State University

Cal Poly Enrollment for Fall 2009



The Meaning and Properties of Fractions

Both the table and pie chart use fractions to specify how the students at Cal Poly are distributed among the different schools within the university.

From the table, we see that $\frac{1}{5}$ (one-fifth) of the students are enrolled in the School of Engineering.

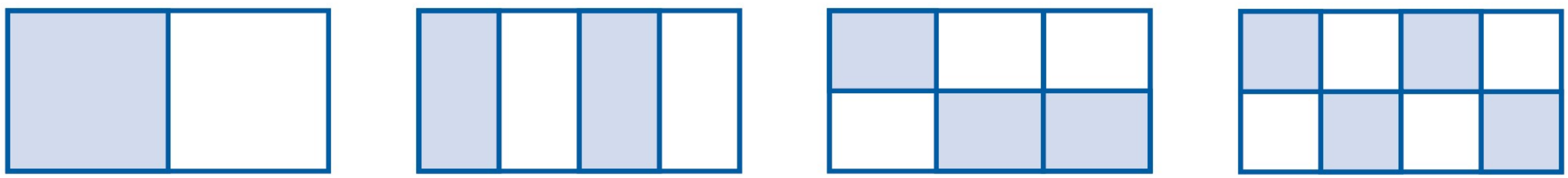
This means that one out of every four students at Cal Poly is studying engineering.

The fraction $\frac{1}{5}$ tells us we have 1 part of 5 equal parts.

The Meaning and Properties of Fractions

Figure 1 below shows a rectangle that has been divided into equal parts in four different ways.

The shaded area for each rectangle is $\frac{1}{2}$ the total area.



a. $\frac{1}{2}$ is shaded **b.** $\frac{2}{4}$ are shaded **c.** $\frac{3}{6}$ are shaded **d.** $\frac{4}{8}$ are shaded

Figure 1

Now that we have an intuitive idea of the meaning of fractions, here are the more formal definitions and vocabulary associated with fractions.

A The Numerator and Denominator

The Numerator and Denominator

Definition

A **fraction** is any number that can be put in the form $\frac{a}{b}$ (also sometimes written a/b), where a and b are numbers and b is not 0.

Some examples of fractions are:

$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{9}{5}$
One-half	*Three-fourths*	*Seven-eighths*	*Nine-fifths*

The Numerator and Denominator

Definition

For the fraction $\frac{a}{b}$, a and b are called the **terms** of the fraction. More specifically, a is called the **numerator,** and b is called the **denominator.**

$$\text{fraction } \frac{a}{b} \begin{matrix} \leftarrow \text{numerator} \\ \leftarrow \text{denominator} \end{matrix}$$

Example 1

The terms of the fraction $\frac{3}{4}$ are 3 and 4. The 3 is called the numerator, and the 4 is called the denominator.

B Proper and Improper Fractions

Proper and Improper Fractions

Now let's more closely examine fractions by defining *proper* and *improper* fractions.

Definition

A **proper fraction** is a fraction in which the numerator is less than the denominator. If the numerator is greater than or equal to the denominator, the fraction is called an **improper fraction.**

Example 4

The fractions $\frac{3}{4}$, $\frac{1}{8}$, and $\frac{9}{10}$ are all proper fractions, because in each case the numerator is less than the denominator.

Example 5

The numbers $\frac{9}{5}$, $\frac{10}{10}$, and 6 are all improper fractions, because in each case the numerator is greater than or equal to the denominator.

(Remember that 6 can be written as $\frac{6}{1}$, in which case 6 is the numerator and 1 is the denominator.)

C Equivalent Fractions

Equivalent Fractions

Some fractions may look different but they still have the same value. To understand this, we must first explore fractions on the number line. We can give meaning to the fraction $\frac{2}{3}$ by using a number line.

If we take that part of the number line from 0 to 1 and divide it into *three equal parts*, we say that we have divided it into *thirds* (see Figure 2).

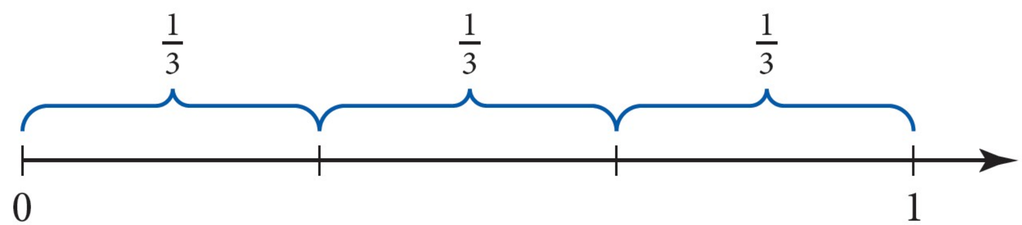


Figure 2

Each of the three segments is $\frac{1}{3}$ (one-third) of the whole segment from 0 to 1.

Equivalent Fractions

Two of these smaller segments together are $\frac{2}{3}$ (two-thirds) of the whole segment.

And three of them would be $\frac{3}{3}$ (three-thirds), or the whole segment, as indicated in Figure 3.

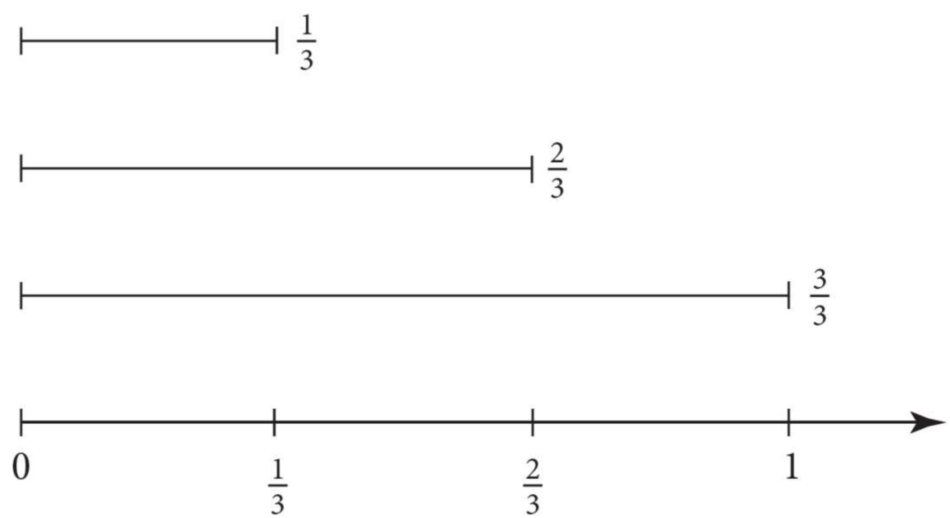


Figure 3

Equivalent Fractions

Let’s do the same thing again with six and twelve equal divisions of the segment from 0 to 1 (see Figure 4).

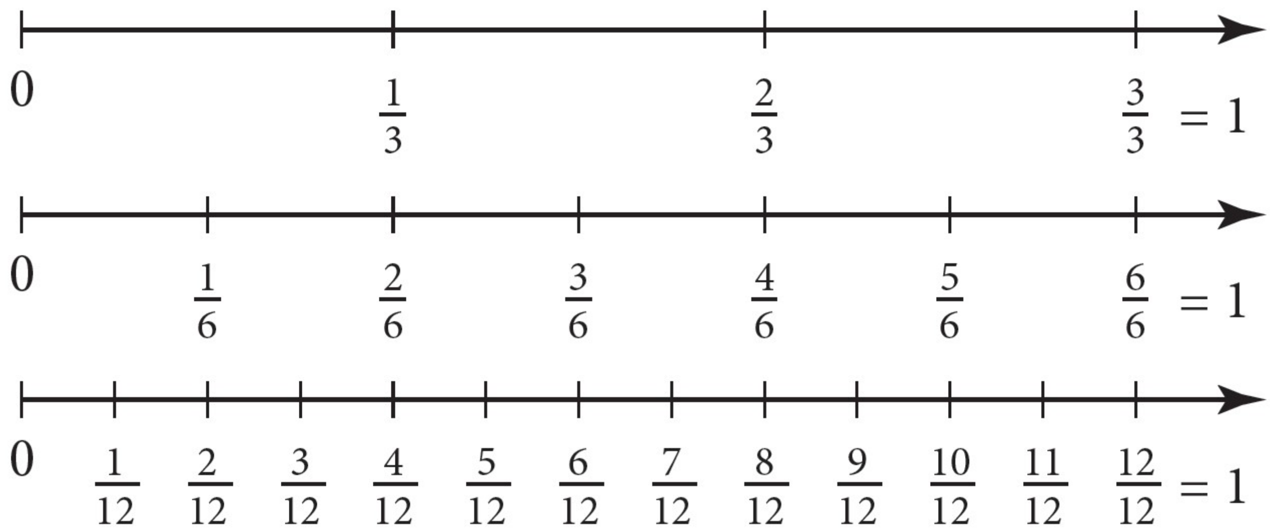


Figure 4

Equivalent Fractions

The same point that we labeled with $\frac{1}{3}$ in Figure 3 is now labeled with $\frac{2}{6}$ and with $\frac{4}{12}$.

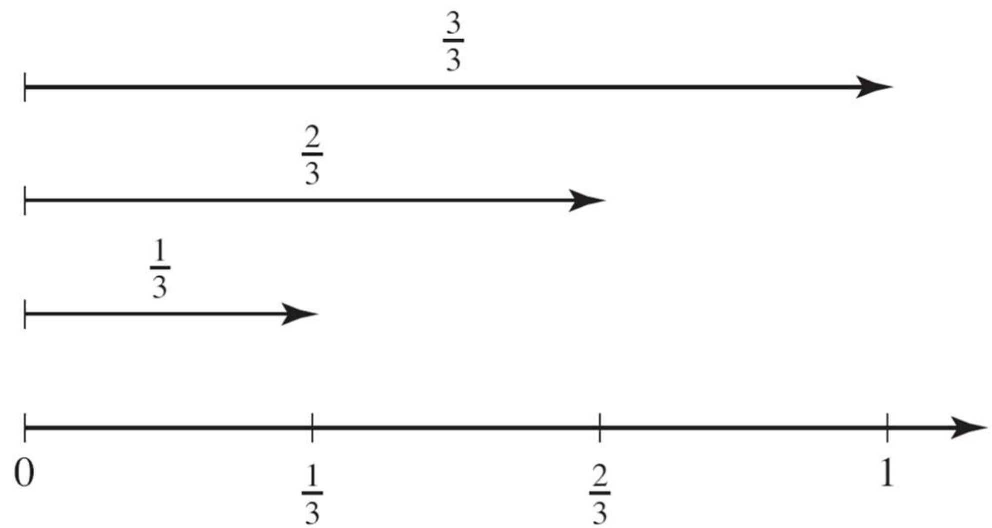


Figure 3

It must be true then that

$$\frac{4}{12} = \frac{2}{6} = \frac{1}{3}$$

Equivalent Fractions

Although these three fractions look different, each names the same point on the number line, as shown in Figure 4.

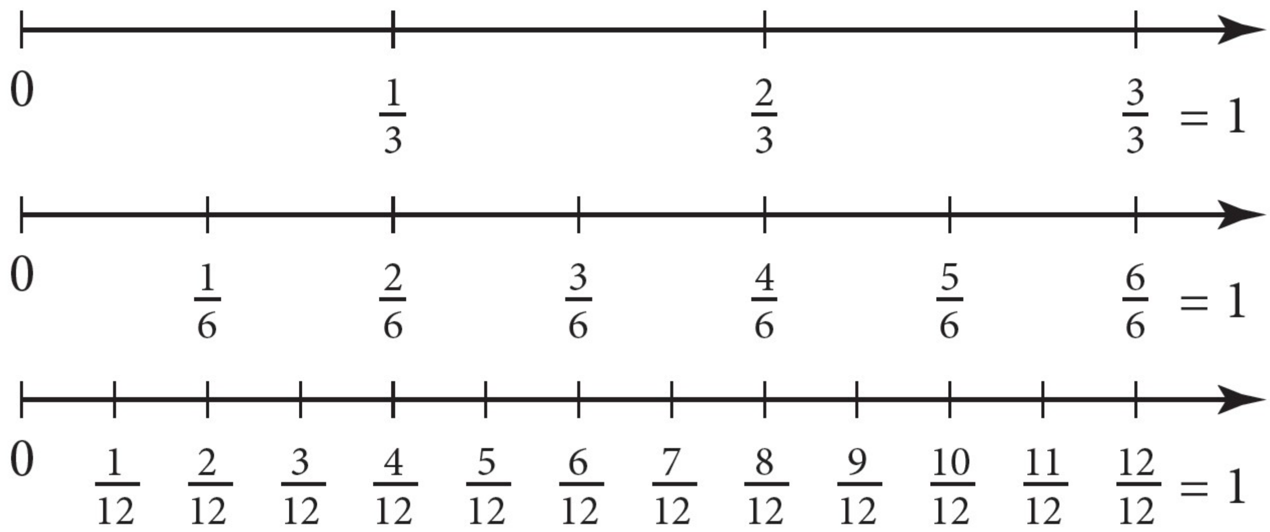


Figure 4

All three fractions have the same value, because they all represent the same number. This also means that these fractions are *equivalent*.

Equivalent Fractions

Definition

Fractions that represent the same number are said to be **equivalent.** Equivalent fractions may look different, but they must have the same value.

It is apparent that every fraction has many different representations, each of which is equivalent to the original fraction.

The next two properties give us a way of changing the terms of a fraction without changing its value.

Equivalent Fractions

Property 1 for Fractions

If a, b, and c are numbers and b and c are not 0, then it is always true that

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$$

In words: If the numerator and the denominator of a fraction are multiplied by the same nonzero number, the resulting fraction is equivalent to the original fraction.

Example 6

Write $\frac{3}{4}$ as an equivalent fraction with denominator 20.

Solution:

The denominator of the original fraction is 4.

The fraction we are trying to find must have a denominator of 20.

We know that if we multiply 4 by 5, we get 20.

Example 6 – *Solution*

cont'd

Property 1 indicates that we are free to multiply the denominator by 5 so long as we do the same to the numerator.

$$\frac{3}{4} = \frac{3 \cdot \mathbf{5}}{4 \cdot \mathbf{5}} = \frac{15}{20} \qquad \text{Multiply numerator and denominator by 5.}$$

The fraction $\frac{15}{20}$ is equivalent to the fraction $\frac{3}{4}$.

Equivalent Fractions

Property 2 for Fractions

If a, b, and c are integers and b and c are not 0, then it is always true that

$$\frac{a}{b} = \frac{a \div c}{b \div c}$$

In words: If the numerator and the denominator of a fraction are divided by the same nonzero number, the resulting fraction is equivalent to the original fraction.

Example 7

Write $\frac{10}{12}$ as an equivalent fraction with denominator 6.

Solution:

If we divide the original denominator 12 by 2, we obtain 6.

Property 2 indicates that if we divide both the numerator and the denominator by 2, the resulting fraction will be equal to the original fraction.

$$\frac{10}{12} = \frac{10 \div \mathbf{2}}{12 \div \mathbf{2}} = \frac{5}{6}$$

D Fractions and Division

Fractions and Division

There are two situations involving fractions and the number 1 that occur frequently in mathematics.

The first is when the denominator of a fraction is 1.

In this case, if we let *a* represent any number, then

$$\frac{a}{1} = a$$

The second situation occurs when the numerator and the denominator of a fraction are the same nonzero number:

$$\frac{a}{a} = 1$$

Example 8

Simplify each expression.

a. $\frac{24}{1}$ **b.** $\frac{24}{24}$ **c.** $\frac{48x}{24x}$ **d.** $\frac{72y}{24y}$

Solution:

In each case we divide the numerator by the denominator.

a. $\frac{24}{1} = 24$ **b.** $\frac{24}{24} = 1$ **c.** $\frac{48x}{24x} = 2$ **d.** $\frac{72y}{24y} = 3$

E Comparing Fractions

Example 9

Write each fraction as an equivalent fraction with denominator 24. Then write them in order from smallest to largest.

$$\frac{5}{8} \qquad \frac{5}{6} \qquad \frac{3}{4} \qquad \frac{2}{3}$$

Solution:

We begin by writing each fraction as an equivalent fraction with denominator 24.

$$\frac{5}{8} = \frac{15}{24} \qquad \frac{5}{6} = \frac{20}{24} \qquad \frac{3}{4} = \frac{18}{24} \qquad \frac{2}{3} = \frac{16}{24}$$

Example 9 – *Solution*

cont'd

Now that they all have the same denominator, the smallest fraction is the one with the smallest numerator and the largest fraction is the one with the largest numerator.

Writing them in order from smallest to largest we have:

$$\frac{15}{24} < \frac{16}{24} < \frac{18}{24} < \frac{20}{24}$$

$$\text{or}$$

$$\frac{5}{8} < \frac{2}{3} < \frac{3}{4} < \frac{5}{6}$$