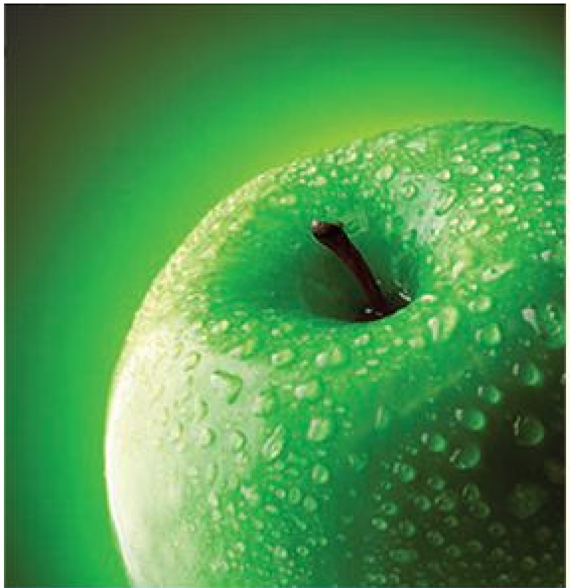


3 Understanding Addition and Subtraction

SECTION 3.3

Understanding Addition and Subtraction of Fractions

What do you think?

- Why *do* we need a common denominator to add (or subtract) two fractions?

- What pictorial representations can we use for addition and subtraction with fractions?

Understanding Addition and Subtraction of Fractions

There are many algorithms that enable us to manipulate fractions quickly: I is simply not enough for an elementary teacher to know how to compute. It is crucial that the teacher also know the *whys* behind the *hows* and can use a variety of representations to explain them.

Investigation 3.3a – *Using Fraction Models to Understand Addition of Fractions*

Using the area model, the length model, or the set model, determine the sum of $\frac{1}{2}+\frac{1}{3}$ and then explain *why* we need to find a common denominator in order to add fractions.

Investigation 3.3a – *Discussion*

This is an excellent place to demonstrate the role of manipulatives.

Using pattern blocks

How might you use pattern blocks to add $\frac{1}{2}$ and $\frac{1}{3}$ (see Figure 3.8)?

Figure 3.8

Investigation 3.3a – *Discussion* continued

If we let the hexagon represent 1, then the trapezoid is $\frac{1}{2}$, the parallelogram is $\frac{1}{3}$, and the triangle is $\frac{1}{6}$. Using the notion of addition as combining, we can combine the two, and we have

$$\frac{1}{2}+\frac{1}{3}=$$

.

We could be humorous and say that parallelogram + trapezoid = baby carriage! The key question, though, is to determine the value of this amount. The solution lies in realizing that to name an amount with a fraction, we must have equal-size parts.

Investigation 3.3a – *Discussion* continued

If you are familiar with pattern blocks, you realize that an equivalent representation is to cover this amount with 5 triangles, as in Figure 3.9.

Figure 3.9

Because the value of each triangle is $\frac{1}{6}$, we can now say that $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.

Investigation 3.3a – *Discussion* continued

This model also helps us see the process for getting the common denominator. Because 3 triangles are the same size as 1 trapezoid, we can see that $\frac{1}{2} = \frac{3}{6}$.

Addition of Fractions

Addition of Fractions

Using Cuisenaire rods

This is a different question when using Cuisenaire rods, because there is no "natural" choice for a unit. Thus, you have to think of a color for which you can represent $\frac{1}{2}$ of that color *and* $\frac{1}{3}$ of that color.

One of many different solutions is to choose the dark green rod to have a value of 1.

Addition of Fractions

The red rod now has a value of $\frac{1}{3}$, and the light green rod has a value of $\frac{1}{2}$ [Figure 3.10(a)].

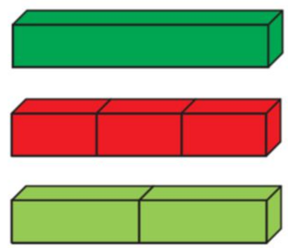


Figure 3.10(a)

As with pattern blocks, when we combine these two parts, in order to name the amount we have to find a way to represent this length with equal-size pieces, and 5 white rods have the same length.

Addition of Fractions

Because 6 white rods have the same length as the dark green rod, each of the white rods has a value of $\frac{1}{6}$, and therefore 5 of them have a value of $\frac{5}{6}$ [Figure 3.10(b)].

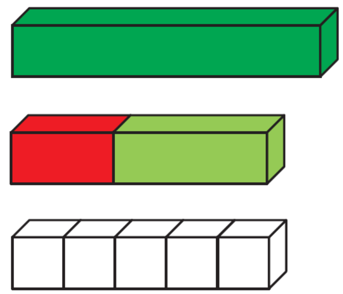


Figure 3.10(b)

How does this model help us see the common denominator process?

Least Common Multiple and Lowest Common Denominator

Least Common Multiple and Lowest Common Denominator

Let us step back for a moment and explore the concept of least common multiple (LCM). What do you think “least common multiple” means, and how does it relate to the “lowest common denominator”?

Let us examine this concept word by word, using a specific example. Suppose we wanted to find the least common multiple of 8 and 12. At the most basic level, we can start listing multiples of 8 and 12 until we find the first multiple they have in common.

Least Common Multiple and Lowest Common Denominator

Multiples of 8 = {8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96, 104,…}

Multiples of 12 = {12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132,…}

What multiples do the two numbers have in common?

These two numbers have many common multiples: {24, 48, 72, 96,…}.

Least Common Multiple and Lowest Common Denominator

Because the least of the common multiples is 24, we say that LCM(8, 12) = 24. This helps us to add fractions with denominators of 8 and 12 because the least common multiple, 24, is also the **lowest common denominator**.

Strategies for Finding the LCM

Strategies for Finding the LCM

Now let us examine how we might find the LCM of 18 and 40. Again, there are many ways to determine the LCM of two numbers. We will examine several ways.

One way of determining the LCM of 18 and 40 is to construct the LCM by beginning with one of the numbers and applying our understanding of LCM.

Strategies for Finding the LCM

This process is illustrated in the following discussion:

Reasoning	**The work**
The LCM *must* contain all the factors of 18.	$18 = 2 \cdot 3 \cdot 3$, LCM(18, 40) must contain: $2 \cdot 3 \cdot 3$
Now, in order *also* to be a multiple of 40, the LCM will have to contain all the actors in the prime factorization of 40.	$40 = 2 \cdot 2 \cdot 2 \cdot 5$

Strategies for Finding the LCM

Looking now at the factors of 18, what factors of 40 are we missing?

We need to put two more 2s and one 5 into our prime factorization of LCM(18, 40).

LCM(18, 40) must contain: **2** · **2** · 2 · 3 · 3 · **5** That is, LCM(18, 40) = 360.

Strategies for Finding the LCM

Another way of illustrating this process is to note the prime factorizations of 18 and 40 and realize that the least common multiple must contain all the factors in both numbers with no redundancies. That is, the LCM needs to contain three 2s, two 3s, and one 5.

$$\begin{array}{ccccccc} & & 2 \cdot & 3 \cdot & 3 & & \\ 2 \cdot & 2 \cdot & 2 \cdot & & & \cdot 5 & \end{array}$$

Using Prime Factorization and Exponents to Find the LCM

Using Prime Factorization and Exponents to Find the LCM

A more formal way to find the LCM is connected to one of the ways in which we find the greatest common factor. This method comes from representing the prime factorization of each number in exponential form:

$$18 = 2 \cdot 3 \cdot 3 = 2^1 \cdot 3^2$$

$$40 = 2 \cdot 2 \cdot 2 \cdot 5 = 2^3 \cdot 5^1$$

When finding the GCF, we took the smaller exponent of all common factors. What do you think we will do when finding the LCM?

Using Prime Factorization and Exponents to Find the LCM

In order for a number to be the LCM, it must contain *all* the factors in numbers. For example, because the prime factorization of 18 contains a 2, the LCM must contain a 2. However, because the prime factorization of 40 contains three 2s, the LCM must contain three 2s.

Thus, when we examine the prime factorization of each number, whenever there is a common factor, we must take the greater exponent.

Using Prime Factorization and Exponents to Find the LCM

Using this method, we find that $\text{LCM}(18, 40) = 2^3 \cdot 3^2 \cdot 5^1$. Do you see why?

The only factor that 18 and 40 have in common is 2, and the greatest exponent above 2 is 3 (meaning that 2 · 2 · 2 is a factor of 40). Therefore, the prime factorization of the LCM must contain 2^3.

The noncommon prime factors are 3 and 5, so 3^2 and 5^1 are also placed in the prime factorization of the LCM.

Using Prime Factorization and Exponents to Find the LCM

Below are two different ways in which the concept of LCM can emerge in elementary school. Teachers may give their students a 100 chart and ask them to draw a circle around multiples of 8 and a square around multiples of 12, as in Table 3.2.

1	2	3	4	5	6	7	8	9	10
11	$\boxed{12}$	13	14	15	16	17	18	19	20
21	22	23	$\boxed{24}$	25	26	27	28	29	30
31	32	33	34	35	$\boxed{36}$	37	38	39	40
41	42	43	44	45	46	47	$\boxed{48}$	49	50

Table 3.2

Using Prime Factorization and Exponents to Find the LCM

When students discover that some numbers have both a circle and a square around them, the teacher can ask, “How would you describe those numbers?”

In this manner, the students have constructed the concept of LCM. Then, when the more formal definition is presented, they already have experience that connects to this idea.

Using Prime Factorization and Exponents to Find the LCM

Cuisenaire rods can also be used to introduce this concept to children. The teacher might ask the students to select a 4 rod and a 6 rod and then ask, “How many different ways can you make a 4 train and a 6 train that have the same length?” (Figure 3.11) Do you see how the answer to this question connects to the LCM concept?

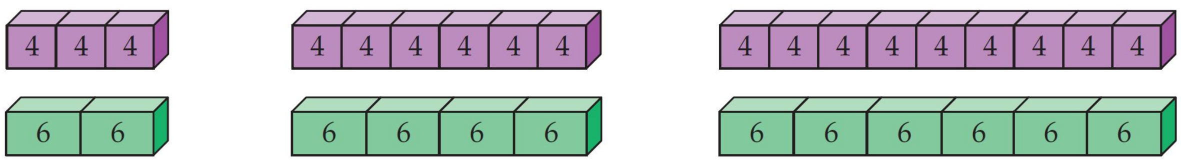


Figure 3.11

Using Prime Factorization and Exponents to Find the LCM

Being able to think about multiples with models helps students develop the ability to find common denominators more easily.

Connecting Concepts to a Procedure for Adding Fractions

Connecting Concepts to a Procedure for Adding Fractions

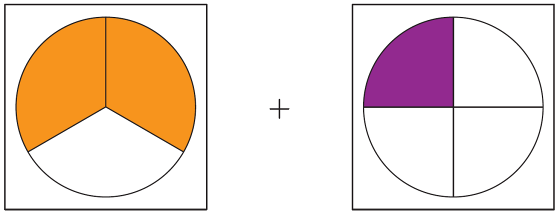
Let us now connect this work to the general procedure for adding fractions. Look at the addition problem below. Try to explain this procedure meaningfully—that is, to *justify* each step. Try using an area model, a linear model, and a set model to help.

$$\frac{2}{3}+\frac{1}{4}=\frac{8}{12}+\frac{3}{12}=\frac{11}{12}$$

A variety of models are helpful in deepening understanding, as well as being able to connect the models to the algorithm. Let's work through each of the three types of models (area, linear, and set) to represent $\frac{2}{3}+\frac{1}{4}$.

Area Model

Let’s use the area of a circle to model this. Look at these pictures and verify that they represent $\frac{2}{3} + \frac{1}{4}$.

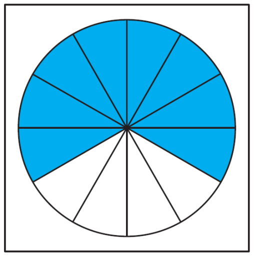


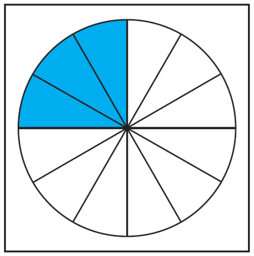
Because the pieces are not the same size, we need to make them the same size. Young students can figure this out by placing pieces on top of the thirds and fourths and learning that the twelfths will fit on both of them.

Connecting Concepts to a Procedure for Adding Fractions

We can also use the fact that we know the LCM(3, 4) = 12. To turn thirds into twelfths, we cover each of the thirds with four of the twelfths, so $\frac{2}{3} = \frac{8}{12}$.

Similarly, $\frac{1}{4} = \frac{3}{12}$ as shown below.



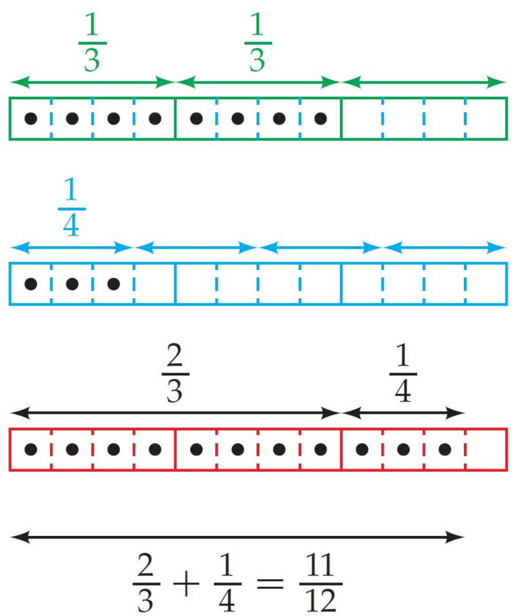


Now we are adding the same size pieces and get $\frac{11}{22}$.

Connecting Concepts to a Procedure for Adding Fractions

Linear Bar Model

Examine this model for $\frac{2}{3}+\frac{1}{4}$ and then read on.



Connecting Concepts to a Procedure for Adding Fractions

We begin with one bar representing the unit, create thirds, and mark two of the thirds. Then we draw a bar of the same size and mark $\frac{1}{4}$.

Again, we have to create the same length bars to be able to add them, so we show that $\frac{2}{3} = \frac{8}{12}$ and $\frac{1}{4} = \frac{3}{12}$. The third bar puts these pieces together to show the answer of $\frac{11}{12}$.

Connecting Concepts to a Procedure for Adding Fractions

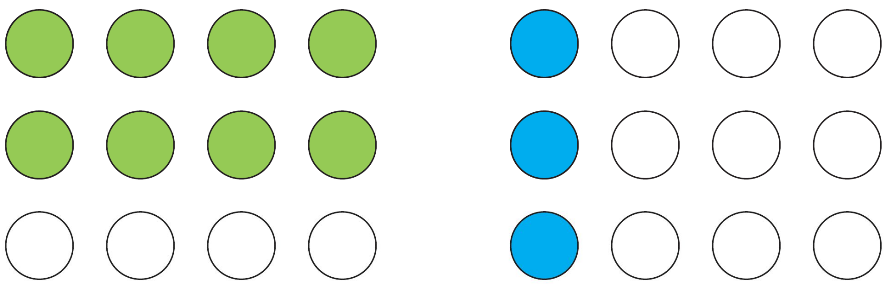
Set Model

Adding fractions with a set model requires that we think ahead to the common denominator to choose how many to put in our set that equals a unit.

Since we know that the LCM of 4 and 3 is 12, we know to draw 12 dots in a 3 by 4 array (since the 3 and 4 are the denominators of the respective fractions).

Connecting Concepts to a Procedure for Adding Fractions

We shade two rows of the first array to show $\frac{2}{3}$ and then we shade 1 column of the second array to show $\frac{1}{4}$. Then, $\frac{2}{3}$ and $\frac{1}{4}$ will look like:



$\frac{2}{3}$ + $\frac{1}{4}$

Putting the shaded objects together, we have 11; since it takes 12 to make 1 unit, this is $\frac{11}{12}$.

Connecting Concepts to a Procedure for Adding Fractions

The procedure is much easier for students to understand and remember when they have these visual models to give meaning to the procedures.

The general procedure used in all of these models is that we determine equivalent fractions by finding a common denominator so that all of the parts are the same size. Then we add the pieces.

Investigation 3.3b – *Using Patterns to Add Fractions*

Let’s say you don’t know how to add fractions with different denominators. Look at the following three examples. Can you see a pattern that would enable you to add other fractions?

$$\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$\frac{1}{5} + \frac{1}{7} = \frac{12}{35}$$

$$\frac{1}{2} + \frac{1}{5} = \frac{7}{10}$$

Investigation 3.3b – *Discussion*

We can describe the pattern in words by saying that whenever you have two fractions with a 1 in the numerator, the numerator of the sum is found by adding the two denominators, and the denominator of the sum is determined by multiplying the two denominators.

Using this pattern, find the sum of

$$\frac{1}{3} + \frac{1}{41}$$

Using the pattern, we would get $\frac{44}{123}$. Using this pattern makes this much simpler than finding a common denominator.

Understanding Subtraction of Fractions

Understanding Subtraction of Fractions

As discussed earlier, subtraction of fractions involves essentially the same processes as does addition. However, many students encounter difficulty in situations that involve regrouping. Therefore, it is important to examine this process also. Consider the following problem: $23\frac{2}{5} - 17\frac{4}{5}$.

First do it on your own, writing down your justification of each step.

$$\begin{array}{r} 23\dfrac{2}{5} = 22\dfrac{7}{5} \\ -17\dfrac{4}{5} = 17\dfrac{4}{5} \\ \hline 5\dfrac{3}{5} \end{array}$$

Understanding Subtraction of Fractions

Let us discuss the transformation from $23\frac{2}{5}$ to $22\frac{7}{5}$. How would you explain this to someone who does not understand it?

We must do some trading, just like in whole-number subtraction, because we cannot subtract $\frac{4}{5}$ from $\frac{2}{5}$.

With fractions, instead of trading 1 ten for 10 ones (as we did with whole number subtraction), we are now trading 1 unit for five fifths (or six sixths, or ten tenths, depending on the common denominator).

Understanding Subtraction of Fractions

Once again, expanded form will help to illustrate the process and justify the equivalence of $23\frac{2}{5}$ and $22\frac{7}{5}$.

$$23\frac{2}{5} = 23 + \frac{2}{5} = (22+1) + \frac{2}{5} = 22 + \left(\frac{5}{5} + \frac{2}{5}\right) = 22\frac{7}{5}$$

Many children and adults do the problem below in the following way:

$$\begin{array}{r} 23\frac{2}{5} = \frac{117}{5} \\ -\,17\frac{4}{5} = \frac{89}{5} \\ \hline \frac{28}{5} = 5\frac{3}{5} \end{array}$$

Investigation 3.3c – *Mental Addition and Subtraction with Fractions*

Because there are many strategies used in mental addition and subtraction of fractions, we will examine them using a process similar to the one we used when developing mental math strategies with whole numbers.

In each of the following problems, try to find the sum or the difference entirely in your head. Briefly describe your strategies.

Briefly describe your strategies.

A.
$$\begin{array}{r} 2\frac{1}{2} \\ +5\frac{3}{4} \\ \hline \end{array}$$

B.
$$\begin{array}{r} 7\frac{3}{4} \\ 5\frac{1}{8} \\ +2\frac{1}{2} \\ \hline \end{array}$$

C.
$$\begin{array}{r} 7 \\ -2\frac{5}{8} \\ \hline \end{array}$$

D.
$$\begin{array}{r} 9\frac{1}{4} \\ -3\frac{5}{8} \\ \hline \end{array}$$

Investigation 3.3c – *Discussion*

Keep the following points in mind for each problem:

- Deciding which strategy is best is often not as helpful as knowing which strategy works best for you. Learning a variety of strategies will help you find strategies that work for you, and that will work for your future students.

- Not all possible useful strategies have been included here, because too many strategies (and 1500-page books) can overwhelm students!

Investigation 3.3c – *Discussion* continued

A. $2\frac{1}{2}+5\frac{3}{4}$

Strategy 1:Use the commutative and associative properties

$$2\frac{1}{2}+5\frac{3}{4}=\left(2+\frac{1}{2}\right)+\left(5+\frac{3}{4}\right)=(2+5)+\left(\frac{1}{2}+\frac{3}{4}\right)=7+\frac{1}{2}+\frac{3}{4}$$

That is, we transform the problem into $7+\frac{1}{2}+\frac{3}{4}$.

Visualize and decompose $\frac{3}{4}$ into $\frac{1}{2}+\frac{1}{4}$; we then have $7+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}=8\frac{1}{4}$.

Investigation 3.3c – *Discussion* continued

Strategy 2: Add up and break into parts

$$\left(2\frac{1}{2}+5\right)+\frac{3}{4}=7\frac{1}{2}+\frac{3}{4}=7\frac{1}{2}+\left(\frac{1}{2}+\frac{1}{4}\right)=8+\frac{1}{4}=8\frac{1}{4}$$

B. $7\frac{3}{4}+5\frac{1}{8}+2\frac{1}{2}$

Strategy 1: Look for compatible numbers

$\frac{3}{4}$ and $\frac{1}{2}$ can be added mentally to get $1\frac{1}{4}$; then add $1\frac{1}{4}+\frac{1}{8}=1\frac{2}{8}+\frac{1}{8}=1\frac{3}{8}$. Add this to the 14, and we have $15\frac{3}{8}$.

Investigation 3.3c – *Discussion* continued

Strategy 2: Convert to common denominator

Because the common denominator is relatively small and the conversions to eighths are relatively easy, some students find it easier to add all three at once, thinking something like this: “6 + 1 + 4 = 11, that means 11 eighths, which is 8 eighths, which equals 1, + 3 eighths.

Add this to the 14, and we have $15\frac{3}{8}$ as the answer.”

Investigation 3.3c – *Discussion* continued

c. $7 - 2\frac{5}{8}$

Strategy 1: Break into parts and subtract

First, 7 − 2 = 5; now take $\frac{5}{8}$ from 5 to get $4\frac{3}{8}$.

Strategy 2: Add up

$2\frac{5}{8} + 4 = 6\frac{5}{8}$. How much more do we need to get to 7? We need $\frac{3}{8}$ more. The answer is $4\frac{3}{8}$.

Investigation 3.3c – *Discussion* continued

D. $9\frac{1}{4} - 3\frac{5}{8}$

Strategy 1:Add up

$3\frac{5}{8}$ plus 5 equals $8\frac{5}{8}$, plus $\frac{3}{8}$ equals 9, plus $\frac{2}{8} = 9\frac{1}{4}$. What you have to remember to add now is $5 + \frac{3}{8} + \frac{2}{8} = 5\frac{5}{8}$.

A number line helps some students to understand the process better (Figure 3.12).

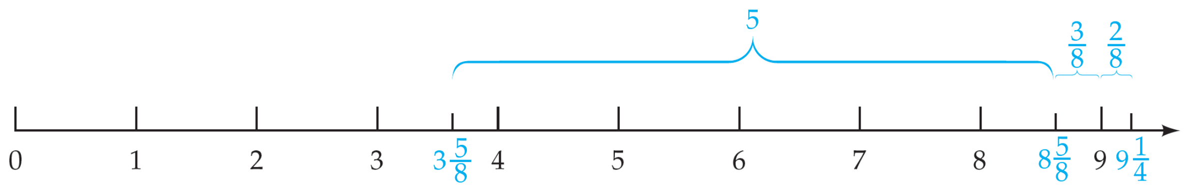


Figure 3.12

Investigation 3.3d – *Estimating Sums and Differences with Fractions*

A. In their retirement, the parents of a close friend of mine have created a dollhouse to represent their dream house. They are shopping, and they see some miniature furniture that might fit in one of the bedrooms. The bed is $\frac{7}{8}$ inch wide; the dresser is $1\frac{1}{2}$ inches long, and the desk is $1\frac{3}{4}$ inches long. If the three articles are placed as shown in Figure 3.13, will they fit into the dollhouse bedroom, which is 4 inches wide?

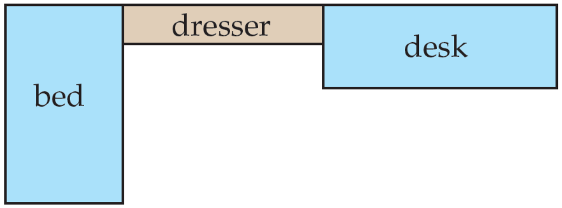


Figure 3.13

Investigation 3.3d – *Discussion*

Strategy 1: Make a quick estimate

A quick estimation reveals that it will be close:

$$\frac{7}{8}+1\frac{1}{2}+1\frac{3}{4}\approx 1+1\frac{1}{2}+1\frac{1}{2}=4$$

Thus, we need a strategy that will be more precise.

Investigation 3.3d – *Discussion* continued

Strategy 2: Add the two easier numbers

$$1\frac{1}{2}+1\frac{3}{4}=1\frac{1}{2}+\left(1\frac{1}{2}+\frac{1}{4}\right)=3\frac{1}{4}$$

We can use number sense here; that is, we know that $3\frac{1}{4}+\frac{3}{4}=4$, and because $\frac{7}{8}>\frac{3}{4}$, we can conclude that the furniture is too big.

B. It rained almost every day this past week. Here are the amounts per day. How much rain fell during the week? First, make a rough estimate. Then make a refined estimate.

Monday	$1\frac{3}{4}$ inches	Friday	0 inches
Tuesday	$\frac{1}{2}$ inch	Saturday	$3\frac{3}{4}$ inches
Wednesday	$\frac{5}{8}$ inch	Sunday	$1\frac{5}{16}$ inches
Thursday	$\frac{7}{8}$ inch		

Investigation 3.3d – *Discussion*

Strategy 1: Make a rough estimate

One way to get a rough estimate is to round each fraction to the nearest $\frac{1}{2}$ inch and keep a cumulative total:

$$2+\frac{1}{2}=\mathbf{2\frac{1}{2}};\ 2\frac{1}{2}\text{ plus }\frac{1}{2}=\mathbf{3};\ 3\text{ plus }1=\mathbf{4};\ 4\text{ plus }4=\mathbf{8};\ 8\text{ plus }1\frac{1}{2}=\mathbf{9\frac{1}{2}}$$

Investigation 3.3d – *Discussion* continued

Strategy 2: Look for compatible numbers

One way to get a closer estimate is to look for compatible numbers—numbers whose sum is close to 1 or numbers that you can quickly add mentally.

$$\underbrace{\left(1\frac{3}{4} + 3\frac{3}{4}\right)}_{5\frac{1}{2}} + \underbrace{\frac{1}{2}}_{\frac{1}{2}} + \underbrace{\left(\frac{5}{8} + 1\frac{5}{16}\right)}_{\text{about } 2} + \underbrace{\frac{7}{8}}_{\text{about } 1} = \text{about } 9$$

Investigation 3.3e – *Using Structure to Do Mental Math*

How could you find the answer to the following without finding a common denominator?

$$1\frac{2}{3}-\frac{1}{4}+2+\frac{1}{3}-\frac{3}{4}$$

Investigation 3.3e – *Discussion*

In the first problem, rearranging the numbers makes this difficult-looking problem quite easy to do in your head. If you look at the parts of this problem, knowing the flexibility to move around, it can be seen as

$$1\frac{2}{3}+\frac{1}{3}-\frac{1}{4}-\frac{3}{4}+2=2-1+2=3$$