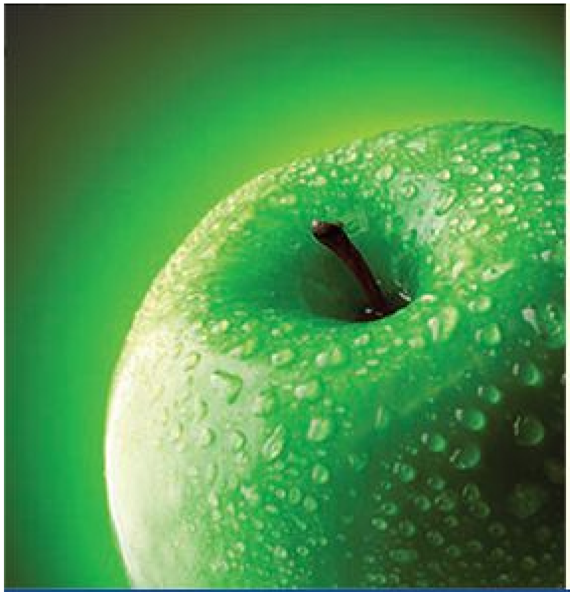


2 The Number System

SECTION 2.3

Decimals, Integers, and Real Numbers

What do you think?

- How would the structure of a decimal type system be different in a base five system?

- Why is there no “oneths” place?

- Why is 28 less than 27?

Decimals, Integers, and Real Numbers

We will consider the rest of the numbers on the number line in this section. Like fractions, decimals are a way to represent parts of a whole. With integers, we move to the left of the zero on the number line.

Decimals

Decimals

Our use of decimal numbers comes out of an alternative way of representing fractions, using the advantages of our base ten system. In fact, the word *decimal* comes from the Latin word *decem*, which means “ten.”

Thus, decimal numbers are any numbers that are written in terms of our base ten place value. We can refer to decimals as fractions whose denominator has been converted into a power of ten.

Decimals

The history of decimals

If you stop to think about it, decimals could not have been invented before our base ten system. Credit for bringing decimals into everyday life goes to Simon Stevin. His book *The Art of Tenths or Decimal Arithmetic* was first published in 1585.

Decimals

The use of decimals was spurred primarily by a desire to make computation easier. It is interesting that even today there is not one universal notation. Some countries use a dot like in the United States, but other countries use a comma to represent a decimal.

For example:

United States, Great Britain, and Australia	Continental Europe and South America
54.23	54,23

Decimals

Much of our use of decimals involves measurement:

- We measure mass: for example, the package weighs 2.4 ounces.
- We measure distance: for example, Joan lives 3.7 miles from school.
- We measure time: for example, the world record for the 100-meter dash is 9.58 seconds.

Decimals

Decimals are used in many aspects of daily life to communicate. For example,

- The U.S. birth rate is 14.1 per 1000 population.
- The average life expectancy in the United States is 77.9 years.
- Interest rates are expressed with decimals, such as 7.5 percent.
- The new city budget is $3.48 million.

Extending Place Value to Parts of a Whole

Extending Place Value to Parts of a Whole

The set of decimals can be seen as an extension of base ten counting numbers.

$$10^0 = 1$$

$$10^{-1} = 1/10 = 0.1$$

$$10^{-2} = 1/10^2 = 1/100 = 0.01$$

thousands	hundreds	tens	ones	tenths	hundredths	thousandths	…
10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	
1000	100	10	1	0.1	0.01	0.001	

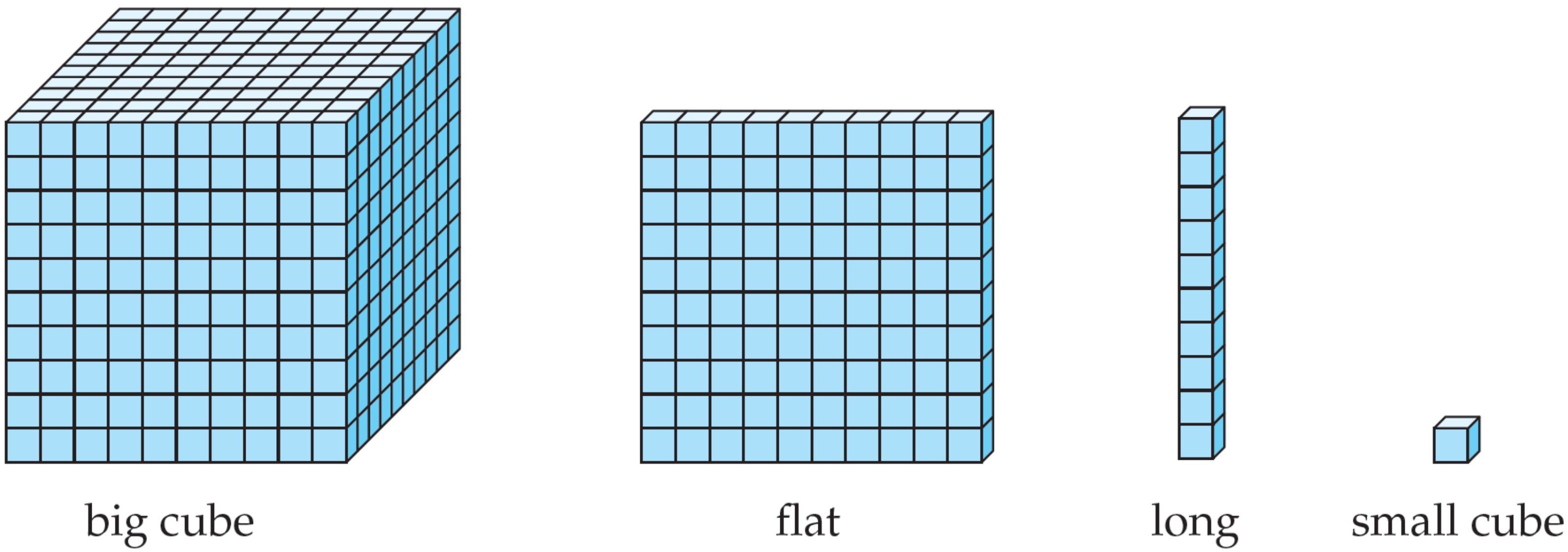
Extending Place Value to Parts of a Whole

That is, the system of decimals and the system of whole numbers are both base ten place value systems. The value of each digit depends on its place, and the value of each place to the right is $\frac{1}{10}$ the value of the previous place. Investigation 2.3a helps to develop your decimal sense and give you a better “feel” for the relative size of tenths, hundredths, thousandths, and so on.

Investigation 2.3a – *Using Base Ten Blocks Strategically*

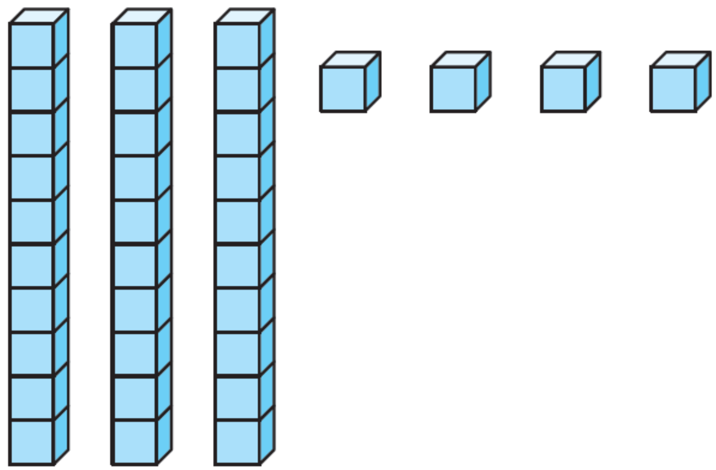
How could you use these base ten blocks to model 34?

How could you use the base ten blocks to model 3.4?



Investigation 2.3a – *Discussion*

To represent 34, let's define the small cube as "1." Then, 34 would look like:



Investigation 2.3a – *Discussion* continued

However, when we try to represent 3.4, we find that using the small cube as “1” will not work since there is no way to represent four-tenths. We need to define one of the larger blocks as “1.” What if we defined the long one to be “1”? What would 3.4 look like?

Three longs then would equal 3 and each small cube would each equal .1, so we would need 4 of those. Therefore, it would be the same picture as above!

Investigation 2.3a – *Discussion* continued

What would 3.4 look like if we defined the big cube as “1”? Then 3 big cubes would model the 3 and 4 flats would equal 4 tenths.

This example illustrates how you as a learner and as a teacher might choose to use these blocks strategically in diverse ways depending on whether you are working with whole numbers or decimals. Mathematical learners (and teachers) must be flexible with how they use tools in order to use them well.

Investigation 2.3b – *Base Ten Blocks and Decimals*

Let's say that a sample of gold weighs 3.6 ounces. How would you use base ten blocks (Figure 2.19) to represent this amount?

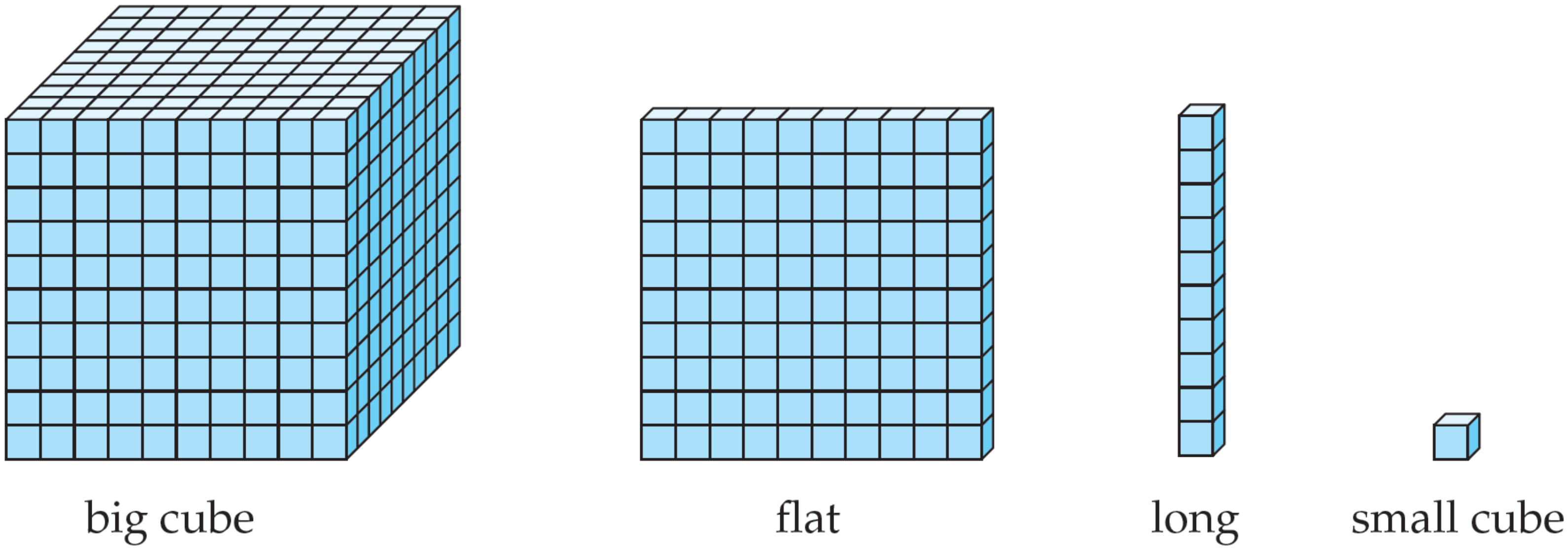


Figure 2.19

Investigation 2.3b – *Discussion*

Any representation requires us to designate one of the pieces as the unit. For example, if we designate the big cube as having a value of 1, then we would represent 3.6 ounces as shown in Figure 2.20. Do you see why?

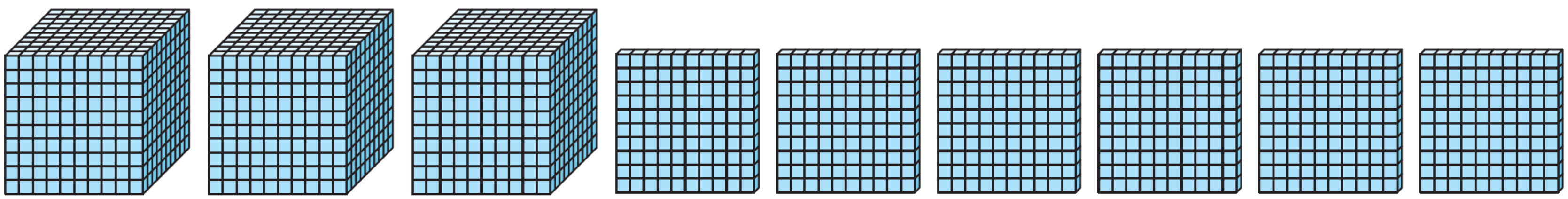


Figure 2.20

However, we could also have selected the flat to be the unit.

Investigation 2.3b – *Discussion* continued

If we had, then we would represent 3.6 ounces as shown in Figure 2.21.

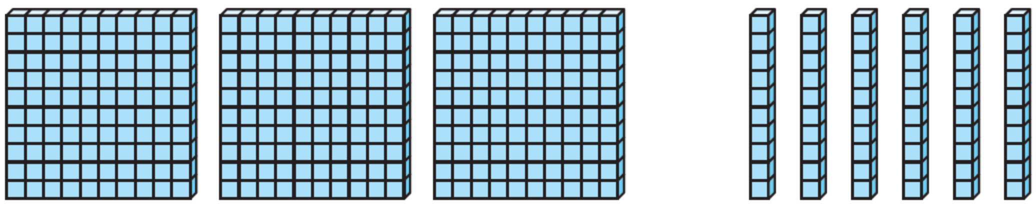


Figure 2.21

The idea of multiple representations connects to our work with whole numbers. For example, just as 24 can be represented by 2 tens and 4 ones or by 24 ones, so too 0.24 can be represented by 2 tenths and 4 hundredths or by 24 hundredths.

Expanded Form

Expanded Form

Like whole numbers, decimal numbers can be expressed in expanded form:

$$624.326$$

$$600 \quad + \; 20 \quad + \; 4 \quad + \; \frac{3}{10} \quad + \; \frac{2}{100} \quad + \; \frac{6}{1000}$$

$$600 \quad + \; 20 \quad + \; 4 \quad + \; .3 \quad + \; .02 \quad + \; .006$$

$$6 \times 10^{2} + 2 \times 10^{1} + 4 \times 10^{0} + 3 \times 10^{-1} + 2 \times 10^{-2} + 6 \times 10^{-3}$$

Expanded Form

As with whole numbers, zeros often prove challenging to make sense of. However, the same principles still apply:

$$2.06$$

$$2 \quad + \frac{0}{10} + \frac{6}{100}$$

$$2 \quad + .06$$

$$2 \times 10^{0} + 6 \times 10^{-2}$$

Zero and Decimals

Zero and Decimals

We will examine the role of zero in decimals in the following two investigations.

Investigation 2.3c – *When Two Decimals Are Equal*

What if we added one or two zeros at the end of 0.2? Would that change its value? That is, do 0.2, 0.20, and 0.200 have the same value? Many sixth- and seventh-graders (and a surprising number of adults) believe that 0.2, 0.20, and 0.200 not only look different but also have different values because of the added zeros. How would you convince such a person that the value is not changed?

Investigation 2.3c – *Discussion*

There are several ways to justify the equality. We will consider two below.

Strategy 1: Connect decimals to fractions

- 0.2 is equivalent to the fraction $\frac{2}{10}$.
- 0.20 is equivalent to the fraction $\frac{20}{100}$, which can be shown to be equivalent to $\frac{2}{10}$.
- 0.200 is equivalent to the fraction $\frac{200}{1000}$, which can be shown to be equivalent to $\frac{2}{10}$.

Investigation 2.3c – *Discussion* continued

Strategy 2: Pictorial

In this case, let a square represent a value of one. Such manipulatives are sold commercially as Decimal Squares. We can divide this square into ten equal pieces, 100 equal pieces, or 1000 equal pieces, as in Figure 2.22.

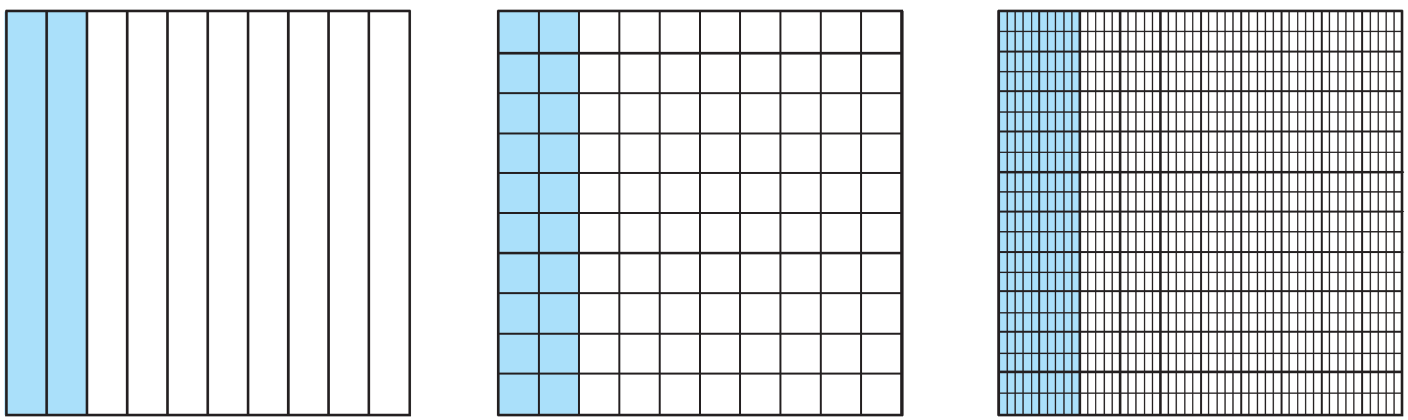


Figure 2.22

Investigation 2.3c – *Discussion* continued

Shade in 0.2 of the first square, 0.20 of the second square, and 0.200 of the third square.

We find that in each case, the same area is shaded. Thus, 0.2, 0.20, and 0.200 are all equivalent decimals.

Investigation 2.3d – *When Is the Zero Necessary and When Is It Optional?*

An important difference between decimals and whole numbers has to do with our old friend zero. Examine each number below. In each case, explain whether you think the zero in the number is necessary, optional, or incorrect. If you think it is necessary, explain why. If you think it is optional, explain why the zero doesn't matter. If you think the use of the zero is incorrect, explain why the zero should not be there.

A. 2.08 **B.** 0.56 **C.** .507 **D.** 20.6 **E.** 3.60

Investigation 2.3d – *Discussion*

A. With 2.08, the zero is necessary: 2 ones, 0 tenths, and 8 hundredths.

B. With 0.56, the use of the zero is optional; it is a convention, like shaking hands with the right hand instead of the left.

Investigation 2.3d – *Discussion* continued

C. With .507, the zero is necessary: 5 tenths, 0 hundredths, and 7 thousandths.

D. With 20.6, the zero is necessary: 2 tens, 0 ones, and 6 tenths.

Investigation 2.3d – *Discussion* continued

E. In one sense, the zero here is optional. Mathematically, 3.653.60; you can verify this using the strategies in Investigation 2.3c. In another sense, however, it depends on how the number is being used.

For example, if we ask how long the room is, what is the difference between a response of 3.6 meters and a response of 3.60 meters?

Investigation 2.3d – *Discussion* continued

A response of 3.6 meters implies that the room has been measured to the nearest tenth of a meter; that is, the length of the room is closer to 3.6 meters than to 3.5 or 3.7 meters.

A response of 3.60 meters implies greater accuracy.

Investigation 2.3e – *Connecting Decimals and Fractions*

As stated before, each decimal can be translated into a fraction. You may be familiar with common translations between decimals and fractions: 0.5 = $\frac{1}{2}$, 0.25 = $\frac{1}{4}$, and so on.

Many problem-solving situations require us to convert a fraction into decimal form or vice versa.

Investigation 2.3e – *Connecting Decimals and Fractions* continued

A. How many of the following common fractions can you convert to decimals immediately? How can you determine the decimal equivalent of others without using a calculator?

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{8} \quad \frac{1}{10} \quad \frac{1}{50} \quad \frac{1}{100} \quad \frac{1}{1000}$$

Investigation 2.3e – *Discussion*

$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{10}$	$\frac{1}{50}$	$\frac{1}{100}$	$\frac{1}{1000}$
$.5$	$.\overline{3}$	$.25$	$.2$	$.1\overline{6}$	$.125$	$.1$	$.02$	$.01$	$.001$

Either by hand or with a calculator, all but $\frac{1}{3}$ and $\frac{1}{6}$ can readily be converted into decimal form. When we convert $\frac{1}{6}$ into a decimal (by dividing one by six), we find that it repeats; that is, the value is 0.1666666. . . . There are two ways to write repeating decimals. We can write $\frac{1}{6}$ in decimal form as 0.166 . . . or as $0.1\overline{6}$. The bar shows the part that repeats. For example, we would write $\frac{348}{999}$ as 0.348348348 . . . or as $0.\overline{348}$.

Investigation 2.3e – *Connecting Decimals and Fractions*

B. Now let us examine translating decimals into fractions.

Investigation 2.3e – *Discussion*

Most people have no trouble with the first two: $\frac{1}{10}$ and $\frac{7}{1000}$. The last decimal is equivalent to $\frac{2}{11}$.

Someone (unknown to history) discovered the process that enables us to conclude that there is a unique fraction associated with every repeating decimal.

Investigation 2.3e – *Discussion*

continued

The process works like this:

Let $x = 0.18181818\ldots$

That is, $x = 0.\overline{18}$

Multiply both sides of this equation by 100: $100x = 18.\overline{18}$

Subtract the first equation from the second: $99x = 18$

Solve for : $x = \frac{18}{99} = \frac{2}{11}$

Decimals, Fractions, and Precision

Decimals, Fractions, and Precision

We have used number lines as a model to represent whole numbers and fractions. The number line can also be used to represent decimals, using the analogy of zooming in. Suppose you had never heard of decimals and had to determine the length of the wire in Figure 2.23 as accurately as you could.

Figure 2.23

Decimals, Fractions, and Precision

A visual inspection of Figure 2.23 shows that the wire appears to be closer to $4\frac{1}{3}$ inches than to $4\frac{1}{3}$ inches. With decimals, we do not look at the most appropriate unit fraction; instead, we look at the fractional length in terms of tenths, hundredths, thousandths, and so forth.

Decimals, Fractions, and Precision

Now imagine zooming in on the wire (Figure 2.24). To the nearest tenth of an inch, how long is the wire?

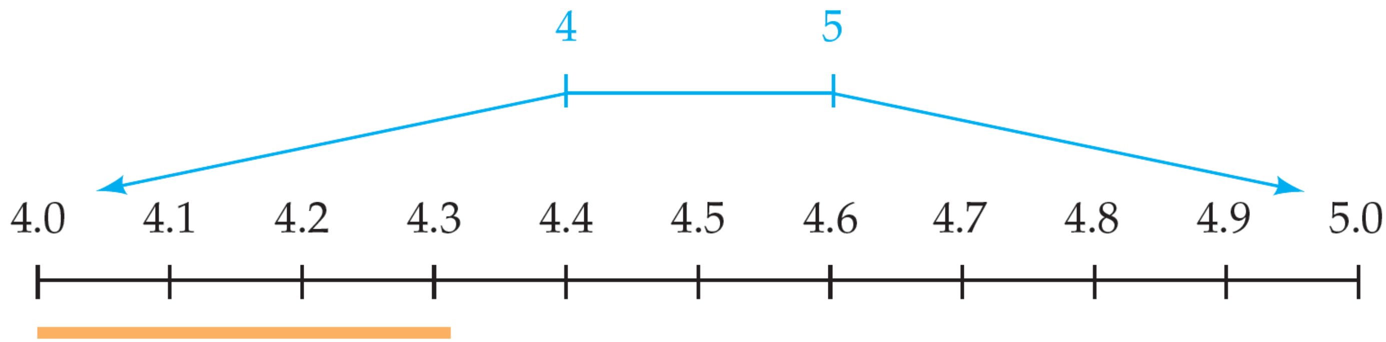


Figure 2.24

Because the length is closer to 4.3 inches than to 4.4 inches, we say that the length is 4.3 inches.

Decimals, Fractions, and Precision

If we want more precision, we can keep zooming in, in which case we move more decimal places to the right. Theoretically, we can continue this magnification process indefinitely. Realistically, electron microscopes are able to measure distances of about one ten-millionth of an inch, or 0.0000001 inch.

An advantage of the metric measuring system is that when reading a metric ruler, the centimeters are divided into tenths (millimeters).

Investigation 2.3f – *Ordering Decimals*

Order the following decimals, from smallest to largest: 0.39, 0.046, and 0.4. Justify your solution.

Investigation 2.3f – *Discussion*

Strategy 1: Use equivalent fractions with common denominators

$$0.39 = \frac{39}{100} = \frac{390}{1000}$$

$$0.046 = \frac{46}{1000}$$

$$0.4 = \frac{4}{10} = \frac{400}{1000}$$

Thus, the order is 0.046, 0.39, 0.4.

Investigation 2.3f – *Discussion* continued

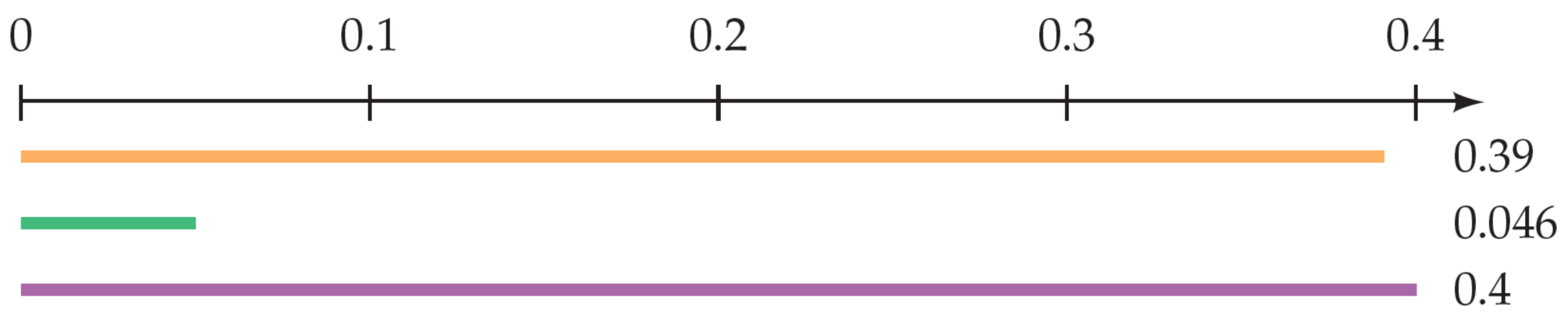
Strategy 2: Line up the decimal points and add zeros

0.39 = 0.390

0.046 = 0.046

0.4 = 0.400

Strategy 3: Represent them on a number line



Investigation 2.3g – *Rounding with Decimals*

The process of rounding with decimals is very similar to the process of rounding with whole numbers. In fact, if you *really* understand place value, then rounding with decimals is very straightforward. Round each of the numbers to the nearest hundredth and to the nearest tenth:

A. 3.623 **B.** 76.199 **C.** 36.215 **D.** 2.0368

E. As you work on these, take mental stock of your confidence level. If you do not feel 100 percent confident, what models (discussed earlier) might you use to increase your confidence?

Investigation 2.3g – *Discussion*

To round to a particular place, we need to look at the digit to the right of the place we are looking for. If that number is less than 5, we keep the number that is there. If the number is 5 or greater, we round up.

A. 3.623 to the nearest hundredth is 3.62, because the 3 to the right of the 2 is less than 5, or in other words, 3.623 is closer to 3.62 than it is to 3.63, and to the nearest tenth is 3.6, because the 2 to the right of the 6 is less than 5, or in other words, 3.623 is closer to 3.6 than to 3.7.

Investigation 2.3g – *Discussion* continued

B. 76.199 to the nearest hundredth is 76.20, and to the nearest tenth is 76.2.

C. 36.215 to the nearest hundredth is 36.22, and to the nearest tenth is 36.2.

D. 2.0368 to the nearest hundredth is 2.04, and to the nearest tenth is 2.0.

Investigation 2.3g – *Discussion* continued

E. Some students find the number line or physical models useful in determining how to round. One other common strategy goes like this: Look at 76.199. If we look at only two decimal places, we have 76.19. Therefore, the question can be seen as: Is 76.199 closer to 76.19 or to 76.20?

Investigation 2.3g – *Discussion* continued

If we insert zeros so that all three decimals are given in thousandths, we have

76.200

76.199

76.190

Do you see how this strategy helps? They are easier to compare with the same number of digits.

Investigation 2.3h – *Decimals and Language*

I saw the following line in a newspaper: “The School Board has proposed a budget of $24.06 million.”

Express the School Board’s proposed budget as a whole number.

Investigation 2.3h – *Discussion*

I have received many different answers from students in my own class, including the following:

24 million 600 thousand dollars

24 million 60 thousand dollars

24 million 6 hundred dollars

24 million and 6 dollars

24 million dollars and 6 cents

Let us look at several strategies that students have used to determine the correct answer.

Investigation 2.3h – *Discussion* continued

Strategy 1: Connect decimals to fractions

Some students who have successfully used this strategy did not know how to represent 24.06 million at first, and so they began with decimal/fraction connections that they did know.

$$24.5 \text{ million} = 24{,}500{,}000 \qquad \text{because } 0.5 = \frac{1}{2}$$

$$24.25 \text{ million} = 24{,}250{,}000 \qquad \text{because } 0.25 = \frac{1}{4}$$

Investigation 2.3h – *Discussion* continued

24.1 million $= 24{,}100{,}000$	because $0.1 = \dfrac{1}{10}$ [This is often a hard place for many students to start, but it makes sense now, in light of the first two steps. Does it make sense to you?]
24.06 million $= 24{,}060{,}000$	because it must be less than $24{,}100{,}000$

Investigation 2.3h – *Discussion* continued

Strategy 2: Connect decimals to expanded form

Some students found the solution by analyzing the meaning of 0.06 million.

24.06 million means 24 million and six hundredths of a million.

Now we have to find the value of six hundredths of a million.

Investigation 2.3h – *Discussion* continued

Using a calculator or computing by hand yields

$0.06 \times 1{,}000{,}000 = 60{,}000$

Thus,

24.06 million = $24{,}000{,}000 + 60{,}000 = 24{,}060{,}000$

Once again we come back to the importance of being able to compose and decompose numbers: whole numbers, integers, fractions, and now decimals.

Investigation 2.3h – *Discussion* continued

Strategy 3: Connect decimals to place value

If your understanding of place value is powerful, you know that the *places* of the digits will not change. Thus, when we represent 24.06 million as a whole number, we simply "add zeros" and we have 24,060,000. The *place* of the 6 does not change!

The language we use when we write checks is connected to place value. How would you write this amount: $206.45?

Investigation 2.3h – *Discussion* continued

It would be incorrect to say “two hundred and six dollars and 45 cents.” Can you see why? To reduce confusion, the convention is to use *and* only once—just before giving the cents. Thus, the conventional way to write \$206.45 is “two hundred six dollars and 45 cents.”

Integers

Integers

Our next extension of the number line is the set of **integers**, which is simply the union of the set of positive integers, the set of **negative integers**, and zero.

Although most of people’s everyday use of mathematics involves positive numbers, we encounter negative numbers in various ways.

Integers

Although elementary students may wonder what is to the left of the zero on a number line and may explore the concept of 223 informally, sixth grade is where students learn about negative numbers, according to the Common Core State Standards.

One of the major goals of this section is for you to understand how the procedures for computing with positive numbers are related to computing when one or more of the numbers are negative.

Integers

We find the earliest mention of negative numbers and how we might compute with them in the works of Brahmagupta (A.D. 628) in India. Other work is found in the writings of al-Khwarizmi (A.D. 825) in Persia and Chu Shi-Ku (A.D. 1300) in China. Cardan (sixteenth century) accepted negative numbers as solutions of equations. However, he referred to them as "false" numbers.

Integers

The term *integer* often causes some confusion for many students because the terms *integer* and *whole number* do not have identical meanings in mathematics and in everyday English.

In mathematics, *integers* refers to the following set of numbers: $\{\ldots -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$. In everyday English, the description of this set is not identical to the description in the first sentence of this section.

Integers

That is, in everyday English, one might say that the integers can be broken down into three sets—negative whole numbers, zero, and positive whole numbers.

However, *whole numbers*, in mathematics, refers to the set {0, 1, 2, 3, . . .}. Thus, saying “negative whole numbers” is like saying “he goed to the store yesterday.” My resolution to this dilemma is to use the terms *positive integers* and *negative integers* but not the terms *positive whole numbers* and *negative whole numbers*.

Integers

Integers in our world

When do we use negative numbers?

People use negative numbers both on and off the job. For instance:

- Businesses use negative numbers to indicate a business loss or “negative profit.”
- We often use negative numbers when describing change. For example, graphs often have negative numbers.
- We use negative numbers to indicate temperatures below zero.

Integers

- Elevations below sea level are often represented with negative numbers.
- A golfer uses negative numbers to indicate a score below par.
- Physicists use negative numbers to indicate negatively charged particles.

Integers

- We use a coordinate system to indicate the position of an object in space. For example, the location of point *P* in Figure 2.25 is (−3, −2).

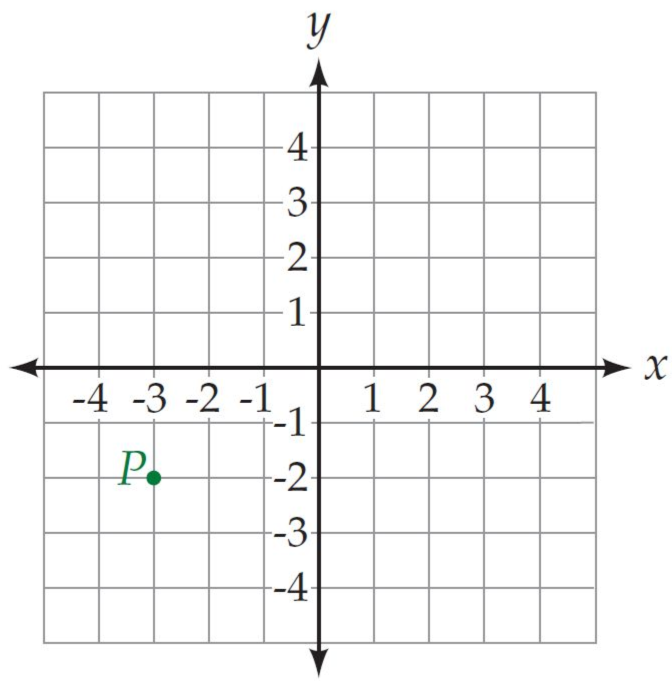


Figure 2.25

Representing Integers

Representing Integers

There are many models for representing integers.

We will focus on the number line model because it is the model to which most real-life applications connect.

Representing Integers

Number lines can be represented horizontally or vertically, as shown in Figure 2.26.

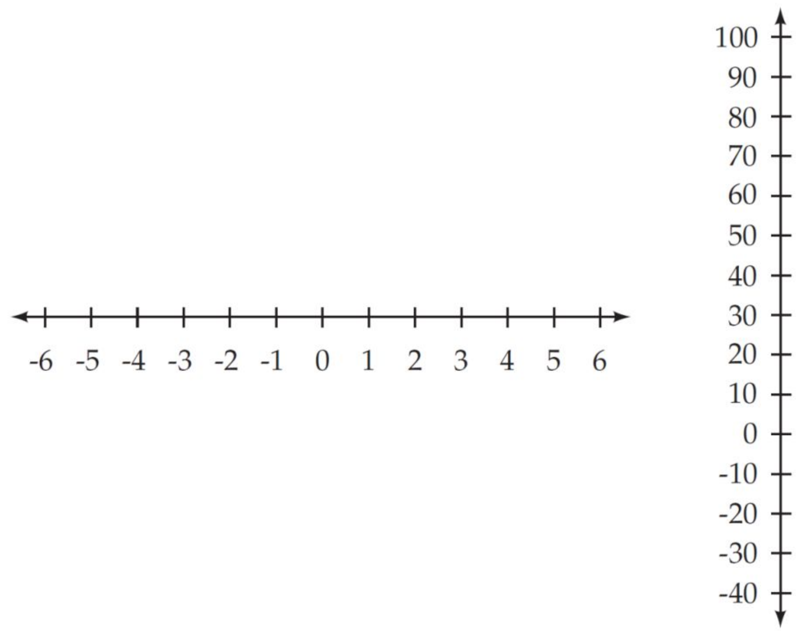


Figure 2.26

Representing Integers

Some other models commonly used to represent integers and their applications include black and red chips, positively and negatively charged particles, and money (assets and liabilities). We will explore these models more when we look at operations with integers in the next two chapters.

Irrational Numbers

Irrational Numbers

Thus far, our investigation of numbers has proceeded from counting numbers to whole numbers to fraction and decimals to integers. There is one more set of numbers that we shall consider in this course. The discovery of this set of numbers is also one of the more dramatic stories in the history of mathematics.

Many important contributions to mathematics came from a group of people who called themselves Pythagoreans.

Irrational Numbers

The Pythagoreans, a community of mathematicians led by Pythagoras, existed in the 500s B.C. Some interesting things about the Pythagoreans were that vegetarianism was strictly followed and that women and men were treated equally in the Pythagorean community.

As you may recall, one of their core beliefs was that all the laws of the universe could be represented using only whole numbers and ratios of whole numbers.

Irrational Numbers

You may recall the Pythagorean Theorem (for any right triangle with legs *a* and *b* and hypotenuse *c*, $a^2 + b^2 = c^2$). The Greeks were the first people to prove this relationship. Ironically, this proof was part of the undoing of the Pythagoreans. Here is how it happened. Consider a right triangle in which the length of each of the two sides is one inch, as shown in Figure 2.27.

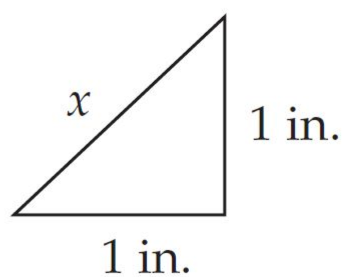


Figure 2.27

Irrational Numbers

If we call the hypotenuse *x* and apply the Pythagorean theorem, we have

$$1^2 + 1^2 = x^2$$

$$x^2 = 2$$

That is, *x* is the number that, when multiplied by itself, equals 2.

Irrational Numbers

This problem was perplexing to the Pythagoreans, because try as they might, they could not find a rational number that, when multiplied by itself, came to *exactly* 2. Using modern notation and base ten arithmetic, we know that we can get *very* close:

$$1.4 \times 1.4 = 1.96$$

$$1.41421 \times 1.41421 = 1.9999899$$

Irrational Numbers

However, there is no rational number (that is, a number that can be expressed as the quotient of two integers) that will produce a product of exactly 2. In modern language, we say $x = \sqrt{2}$ (square root of 2).

Finally, one member of the sect proved that there is no rational number that will solve this problem. This discovery was like a child finding out that Santa Claus is not real or an adult finding out that there really are people from other planets.

Irrational Numbers

What it meant for the Pythagoreans was that one of their cornerstone beliefs—that all the laws of the universe could be represented as whole numbers and ratios of whole numbers—was not true.

What happened to the person who discovered the proof? Legend has it that he was sent out to sea in a leaky rowboat! This is possibly the source of the saying, “Don’t kill the messenger who bears the bad news.” It is also interesting to note that the word *irrational*, even in our present time, means “contrary to reason.”

The Real-Number System

The Real-Number System

With the set of irrational numbers, we can now extend our system of numbers to the set of **real numbers**, which is defined as the union of the sets of rational and irrational numbers.

The next major expansion, which students encounter in high school, is imaginary and complex numbers—for example, the square root of −1.

The Real-Number System

Let us look back at the numbers we have examined in this course with respect to how they are related to each other. We began our study of numbers with the set of natural numbers, which is the first set of numbers young children encounter.

Our first extension of the number line was to add zero. The union of the set of natural numbers and zero is called the set of whole numbers.

The Real-Number System

We then examined two kinds of numbers that children will encounter in elementary school: integers and rational numbers.

Finally, we encountered numbers that cannot be represented as the ratio of two whole numbers—that is, irrational numbers.

The Real-Number System

Figure 2.28 is a visual representation of these different sets of numbers and their relationships.

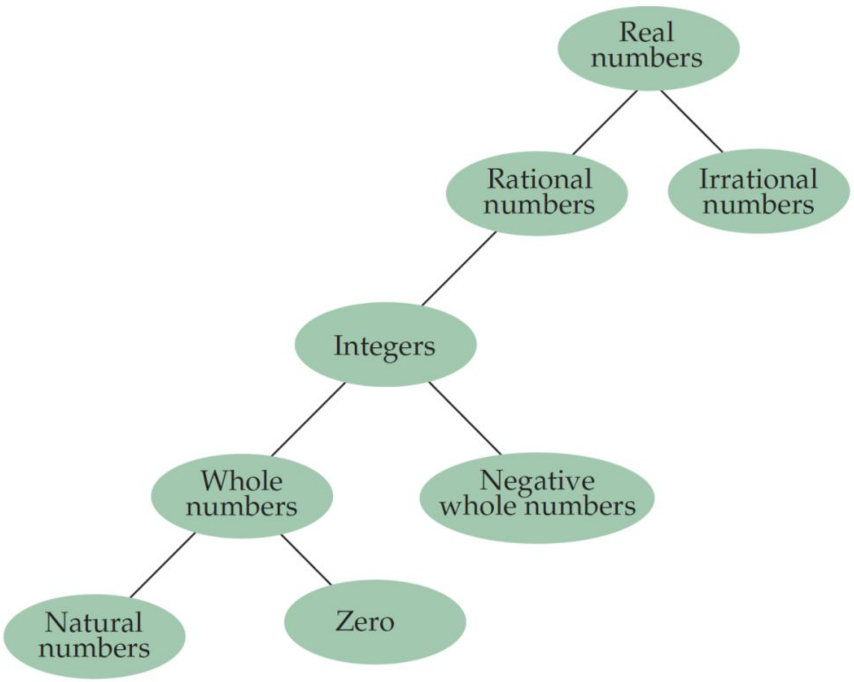


Figure 2.28