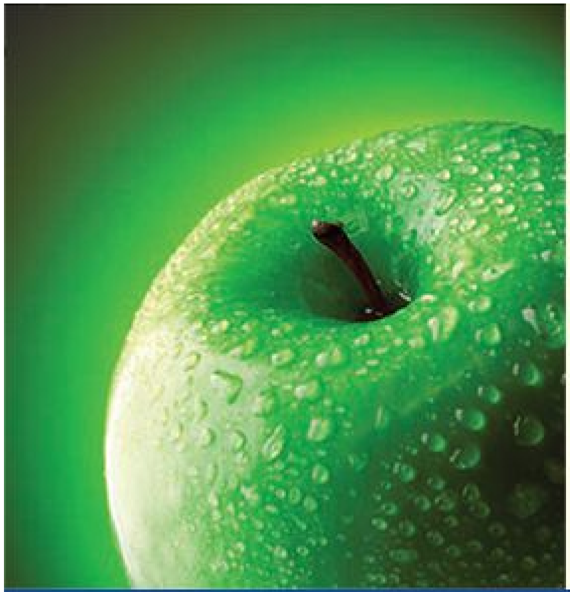


3 Understanding Addition and Subtraction

SECTION

3.4

Understanding Addition and Subtraction of Decimals and Integers

What do you think?

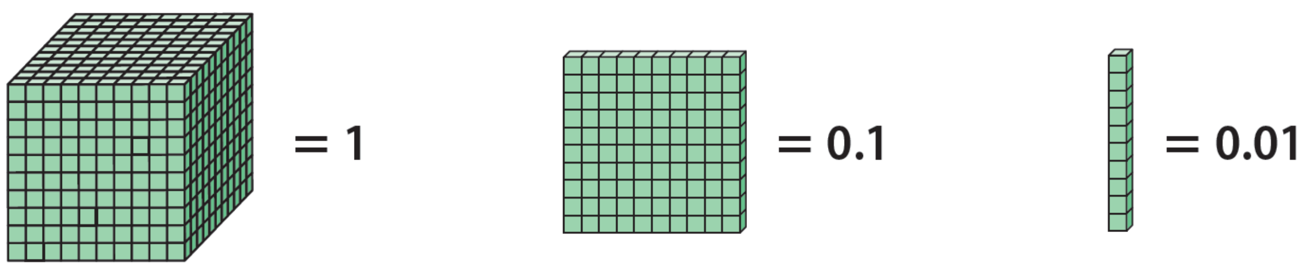
- What are the similarities between adding decimal numbers and whole numbers?

- Why does a negative number plus a positive number sometimes give a positive answer, and sometimes a negative answer?

Understanding Addition and Subtraction of Decimals and Integers

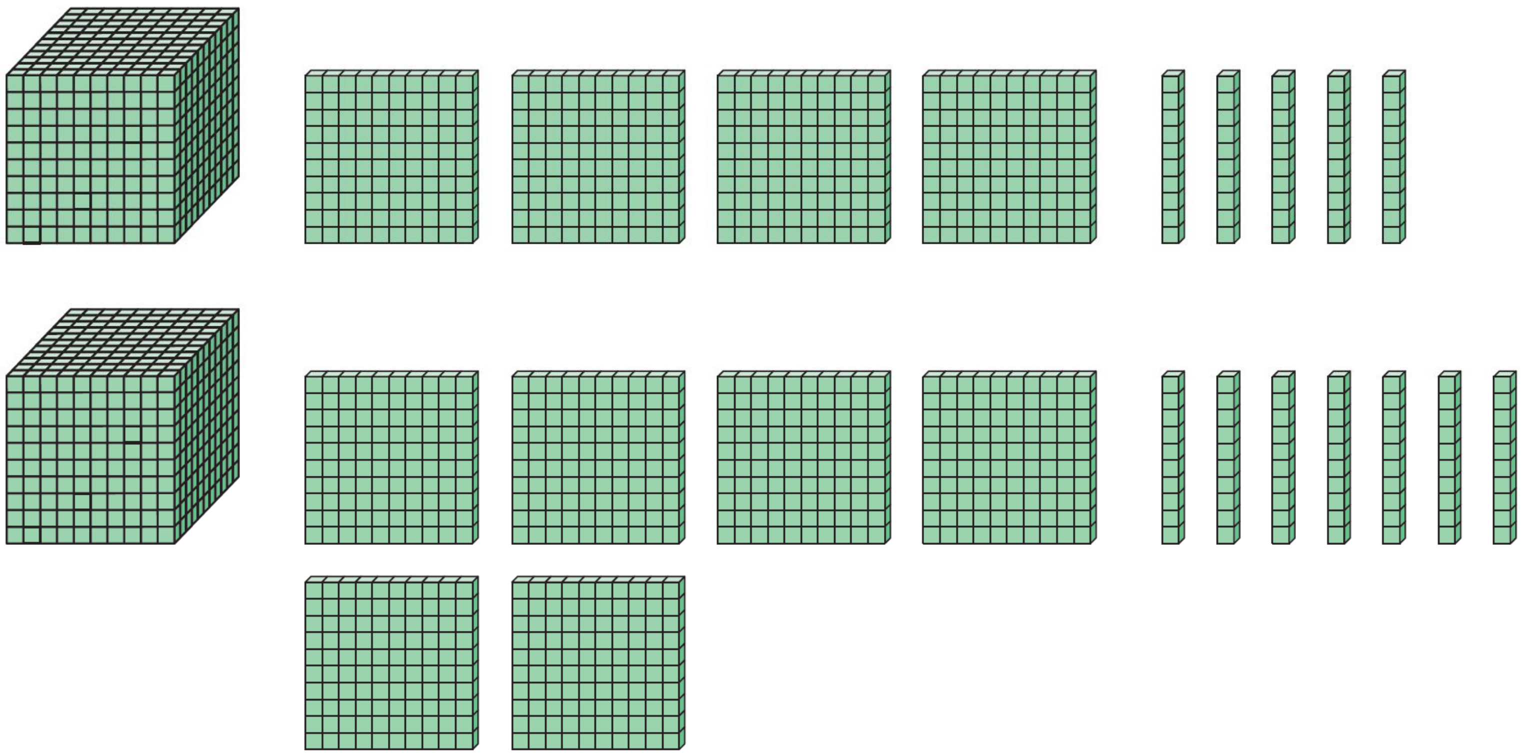
Adding and subtracting decimals works pretty much the same way as adding and subtracting whole numbers.

If we let the large cube represent 1 unit, then a flat would equal 0.1, since a flat is $\frac{1}{10}$ of a large cube. A long would represent 0.01, since a long is $\frac{1}{100}$ of a large cube (that is it takes 100 longs to make a large cube).



Understanding Addition and Subtraction of Decimals and Integers

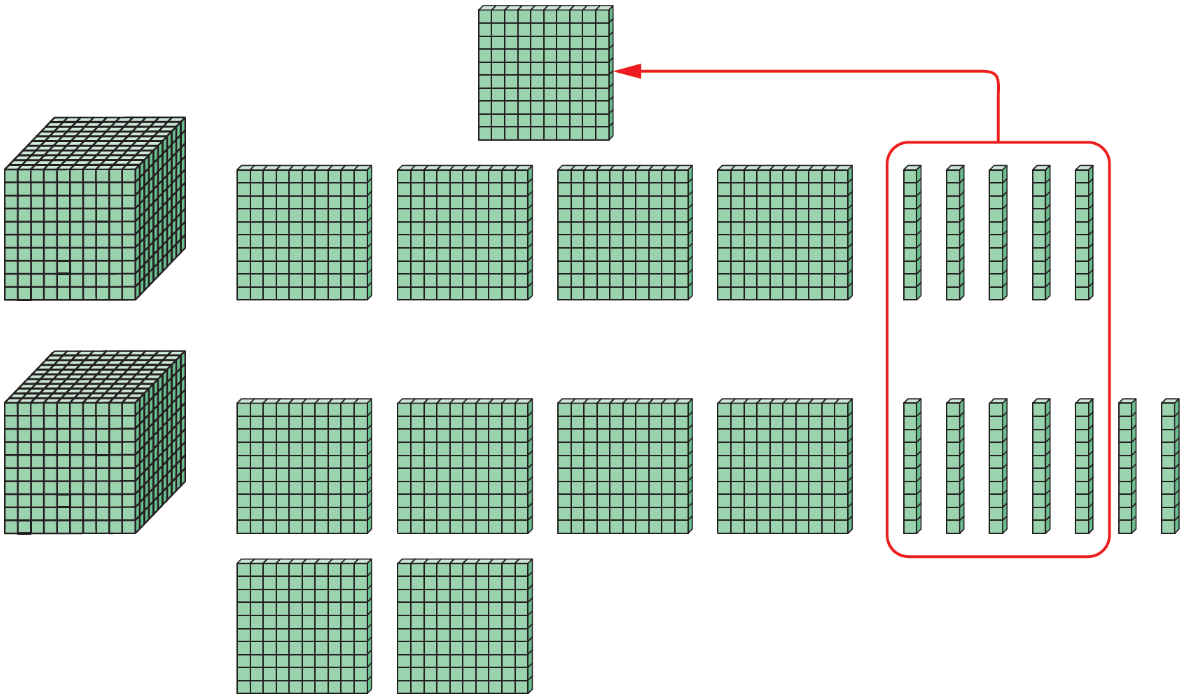
So, using this model,



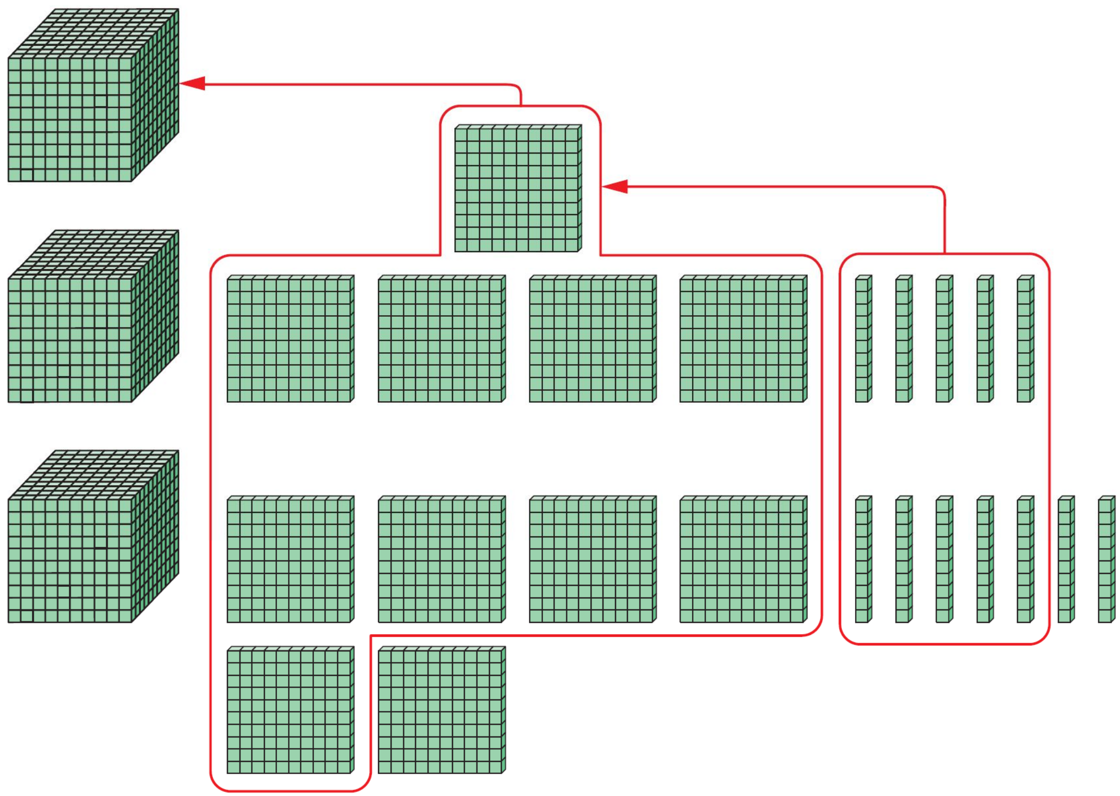
1.45

+1.67

When we add the 5 hundredths with the 7 hundredths, we trade ten of hundredths for one tenth (since $\frac{10}{100} = \frac{1}{10}$) and have 2 hundredths left in that place.



Then, we have the 4 tenths plus the 6 tenths, plus the 1 that we moved over there, then we have 11 tenths, so we trade 10 of these for 1 unit.

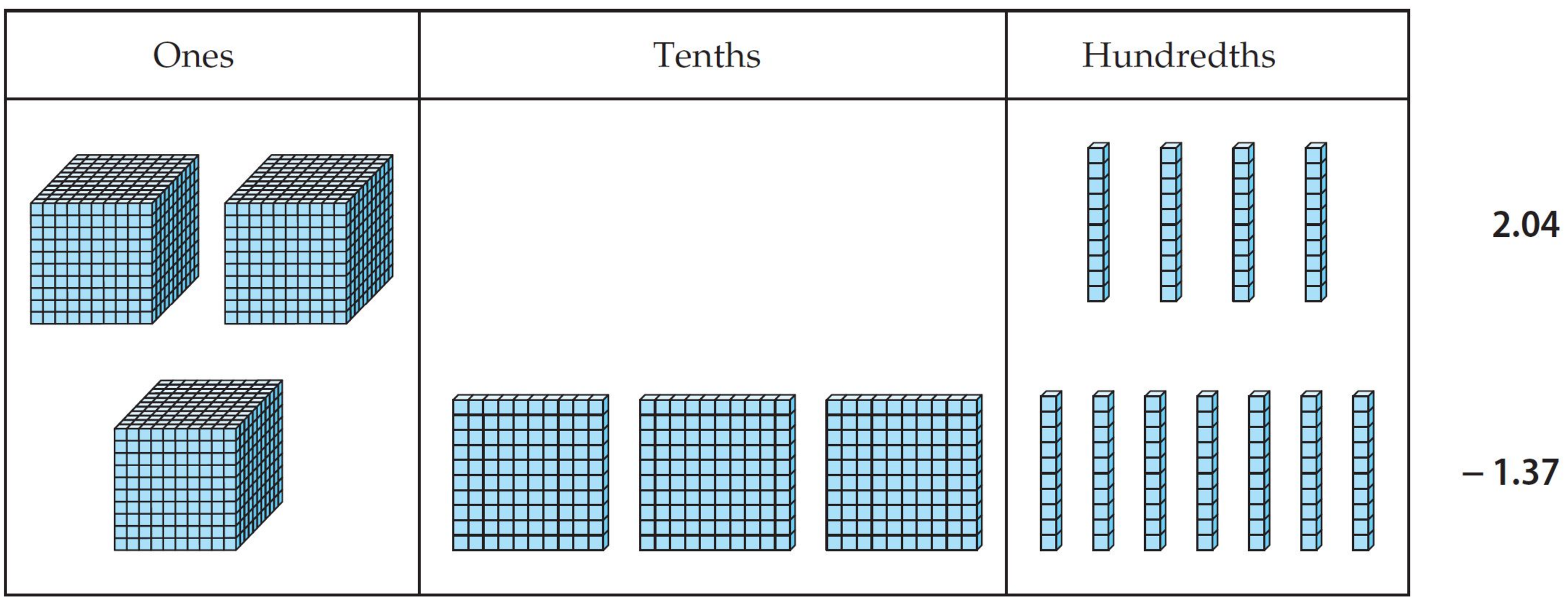


Now, we have 3 units, 1 tenth, and 2 hundredths as the answer, which we write as 3.12.

Addition and Subtraction with Decimals

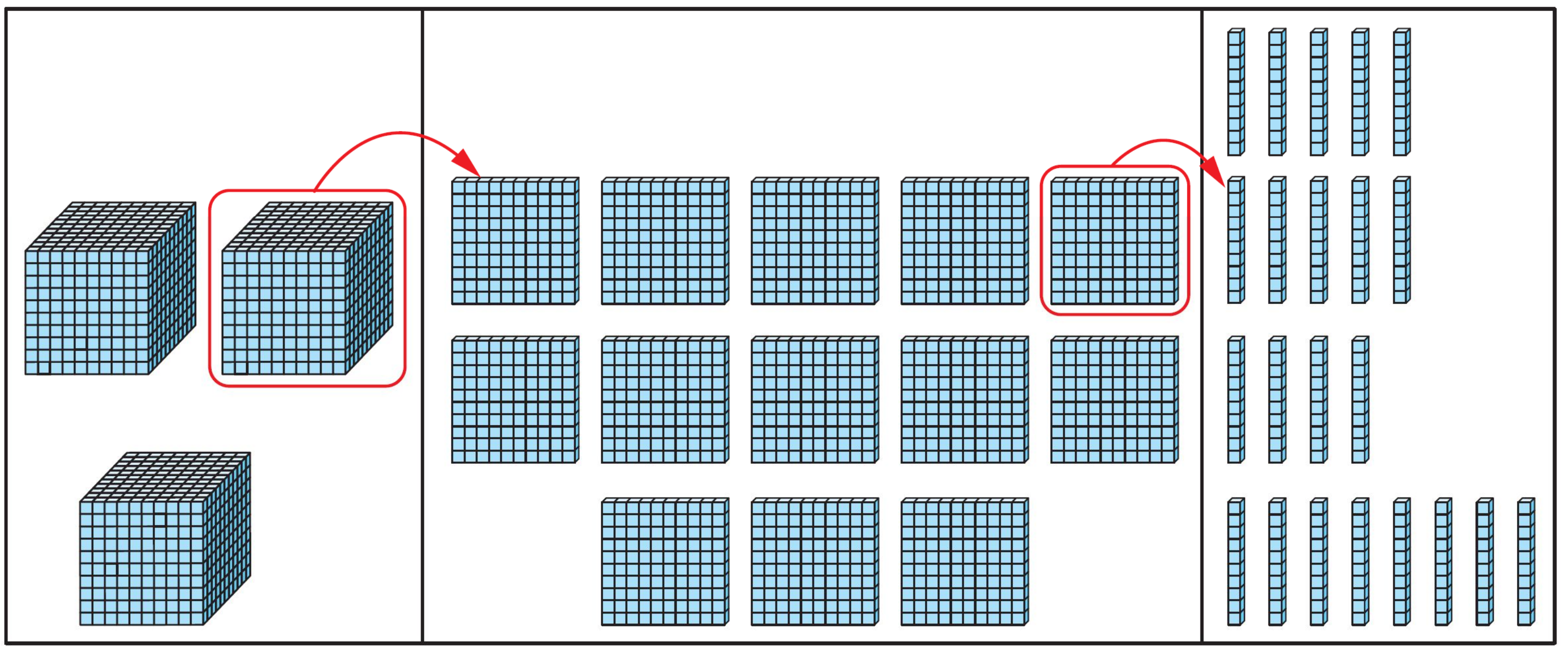
Addition and Subtraction with Decimals

Subtraction with decimals is modeled very similarly. How would you use base ten blocks to represent 2.04 − 1.37? Using the same representations as used in the addition problem:



Addition and Subtraction with Decimals

We want to take away 7 hundredths from 4 hundredths, which means we need to trade. Since there are 0 tenths in 2.04, we have to trade a unit cube into tenths first. Then we can trade 1 tenth for 10 hundredths as shown:



Addition and Subtraction with Decimals

Now we can take away 7 hundredths from the 14 hundredths that are now there to get 7 hundredths, and then take the 3 tenths from the 9 tenths to get 6 tenths, and finally the 1 unit from the 1 unit. Here is the abstract algorithm that matches the pictures:

$$\begin{array}{r} \overset{1}{\cancel{2}}\,.\overset{9}{\cancel{0}}\,{}^{1}4 \\ -\;1\,.3\;7 \\ \hline 0\,.6\;7 \end{array}$$

Addition and Subtraction with Integers

Addition and Subtraction with Integers

We have explored models for addition and subtraction, and now we will explore which of these make more sense with negative numbers.

- addition: combine, increase
- subtraction: take away, comparison, missing addend

As we strive to develop ways to operate *meaningfully* with negative numbers, we will also examine which of these models for the operations work and which do not.

Understanding Addition with Integers

Understanding Addition with Integers

We will use a problem-solving tool to develop algorithms for adding integers: We will examine all possible combinations (cases) and then look for patterns. The four cases are represented below:

	a	*b*	**Example**
Case 1: Both numbers are positive.	+	+	$3+4$
Case 2: One number is positive, one number is negative, and the magnitude of the negative number is greater than the magnitude of the positive number.	+	−	$3+(-4)$
Case 3: One number is positive, one number is negative, and the magnitude of the positive number is greater than the magnitude of the negative number.	−	+	$-3+4$
Case 4: Both numbers are negative.	−	−	$-3+(-4)$

Understanding Addition with Integers

Language

Sometimes the + and − signs are raised as superscripts when indicating a positive and negative number. We have kept them as regular text here, but we need to be careful of language.

For example, we will read the problem −6 − (−8) as "*negative* 6 *minus negative* 8" instead of "minus 6 minus minus 8."

Understanding Addition with Integers

The words *plus* and *minus* will be used to refer to the *operations* of addition and subtraction. The words *positive* and *negative* will be used to refer to the *value* of the number. For practice, translate each of the equations below into words. Check your translations with the sentences at the right.

$5 - (-4) = 9$ 5 minus negative 4 is equal to positive 9.

$32 + (-48) = -16$ 32 plus negative 48 is equal to negative 16.

Understanding Addition with Integers

Let us now examine the first two cases.

Case 1

Both numbers are positive. We have explored this case earlier and drew many models for it like the number line shown in Figure 3.14.

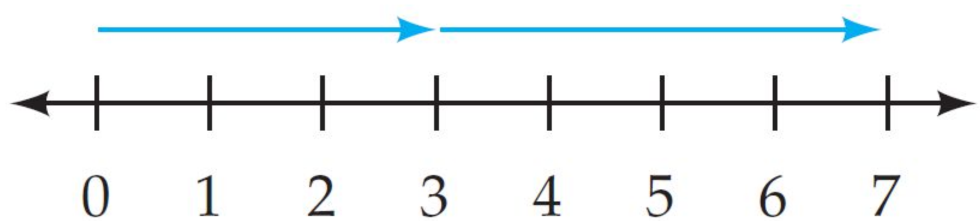


Figure 3.14

Understanding Addition with Integers

Because money can be a great way to help us understand adding and subtracting with negative and positive numbers, we can think of 3 + 4 as if we have $3 in an account and we add $4, then we have $7 in the account.

Understanding Addition with Integers

Case 2

One number is positive, one number is negative, and the magnitude of the negative number is greater than the magnitude of the positive number.

Try several other examples and then try to express a rule that would apply to any addition problem in which the magnitude of the negative number is greater than the magnitude of the positive number.

Understanding Addition with Integers

We will examine a specific problem in detail and then look at generalizations. How would you represent the problem
$3 + (-4)$?

Number line model If a positive number is represented by an arrow pointing to the right (the positive direction), then a negative number is represented by an arrow pointing to the left (the negative direction).

Understanding Addition with Integers

Thus, the problem 3 + (−4) can be represented as shown here.

Understanding Addition with Integers

That is, we begin at zero and move 3 units to the right and then move 4 units to the left, thus, we find that
$3 + (-4) = -1$.

If we use money to help us think about this, we are starting with \$3 in the account. Adding −\$4 is saying a debit of \$4, so if we take out \$4, then there is −\$1 in the account. In other words, it is \$1 overdrawn

Understanding Addition with Integers

As you may have already found, sometimes the sum of a positive number and a negative number is positive (for example, $12 + (-5) = +7$) and sometimes it is negative, as in $3 + (-4) = -1$.

If you do a number of problems, you realize that the procedure for adding a positive number and a negative number feels a lot like subtraction.

Understanding Addition with Integers

In fact, one description of a general rule is: Disregard the signs and subtract the *smaller* number from the *larger* number; the sign of the sum is the same as the sign of the *larger* number. Look at the number line models again to make sense of this rule visually.

Absolute Value

Absolute Value

There is a mathematical way to say "disregard the sign in front of the number," and that is to use the term *absolute value*. Let us first define this concept and then use it to state more precisely the rule for adding a positive number and a negative number.

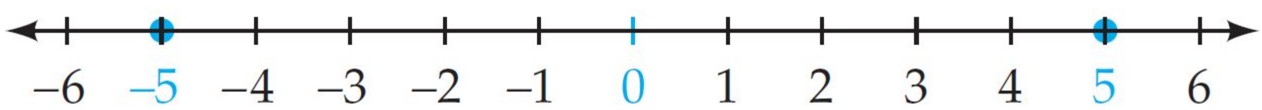
Absolute Value

One way to illustrate the concept of absolute value is to consider how far the number is from zero (the origin).

The **absolute value** tells us the distance of the number from zero.

Absolute Value

For example, the numbers +5 and −5 are both the same distance from zero, so they both have the same absolute value, which is 5 (Figure 3.15).



(Figure 3.15)

Absolute Value

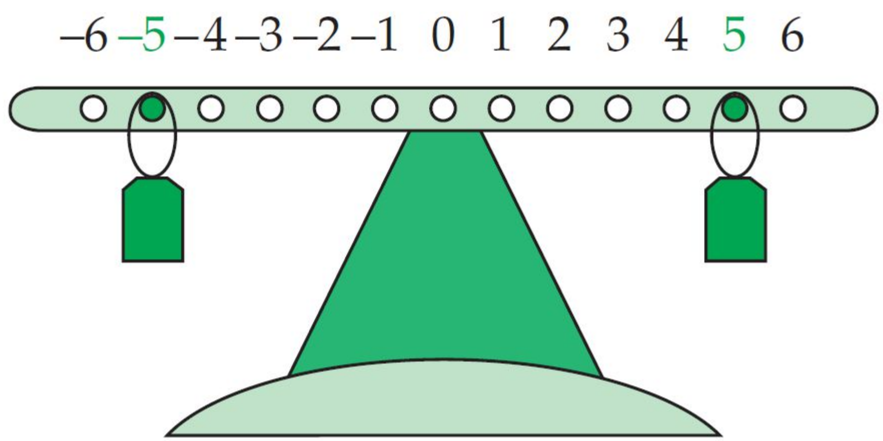
With notation, we say, $|-5| = 5$. Similarly, $|+5| = 5$.

In words, we say that the absolute value of negative 5 is 5, and the absolute value of positive 5 is also 5.

We can also use a balance scale (a common manipulative used in elementary schools) to illustrate the concept of absolute value.

Absolute Value

If we place a weight under −5 and an equal weight under +5, the scale will balance (Figure 3.16).



(Figure 3.16)

Absolute Value

There is also language for referring to pairs of numbers whose absolute values are equal: we say they are opposites or negatives of each other.

Thus, the **opposite** of +6 is −6, and the opposite of −6 is +6. Using another meaning of **negative**, we say that the negative of +6 is −6 and the negative of −6 is +6.

Absolute Value

As you can readily see, when we add any integer and its opposite, the result is zero; that is,

$$a + (-a) = 0$$

Thus we can say that every integer has an additive inverse.

Absolute Value

In this text, we will use the term **additive inverse** instead of *negative* for two reasons. First, the term *negative* often creates a false impression.

For example, if $x = -6$, then $-x = +6$. In other words, when we are working with variables, the value of $-x$ is often positive. Second, when working with multiplication of fractions, we will develop a similar concept with a similar term: the *multiplicative inverse*.

Absolute Value

The concept of absolute value lets us now state more precisely the rule for adding a positive number and a negative number.

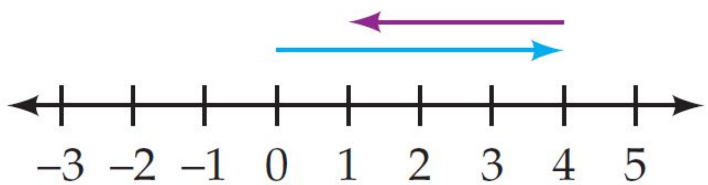
We first find the difference of the absolute values of the two numbers; the sign of the sum is the sign of the number with the larger absolute value.

The Other Two Cases of Integer Addition

The Other Two Cases of Integer Addition

Case 3

One number is positive, one number is negative, and the magnitude of the positive number is greater than the magnitude of the negative number (an example is 4 + (−3)).



The Other Two Cases of Integer Addition

We can use a number line to model this. Here, we draw an arrow from 0 to +4. Since we are adding −3, we draw the arrow back to the left 3 places, which puts us at +1.

If we think of this in the context of money, then we start with $4 in the account and we add a debit of −$3, so we would have +$1 left in the account.

The Other Two Cases of Integer Addition

When we look carefully at all integer addition problems that fall in this category, we find that the generalization stated in the preceding paragraph applies to this case too.

In other words, the similarity between Case 2 problems ($3 + (-4)$, $2 + (-8)$, $-7 + 3$, $-9 + 4$) and Case 3 problems ($4 + (-3)$, $7 + (-2)$, $-6 + 9$, $-1 + 5$) is that one number is positive and one number is negative. Regardless of which number has the larger magnitude, we can use the same procedure to determine the sum.

The Other Two Cases of Integer Addition

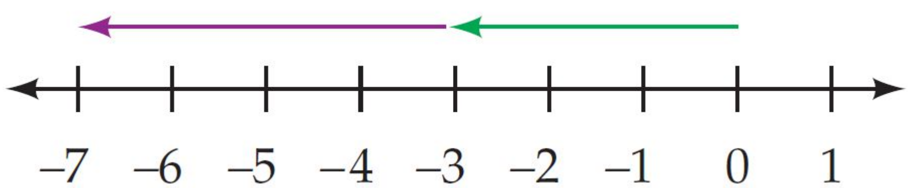
When we first examine all the possibilities when adding integers, we have four distinct subsets.

However, when we look closely at two of these subsets, we find that their similarity (one positive, one negative) outweighs their difference (which number has the greater absolute value).

The Other Two Cases of Integer Addition

Case 4

In Case 4, both numbers are negative. Let us examine a specific problem in detail and then look for generalizations. Figure 3.17 shows a representation of −3 + (−4) on a number line.



(Figure 3.17)

The Other Two Cases of Integer Addition

With the concept of absolute value, we can state the rule for adding two negative numbers precisely: to find the sum of two negative numbers, we first find the sum of the absolute values of the two numbers and then place a negative sign in front of this sum.

A Set Model for Integer Addition

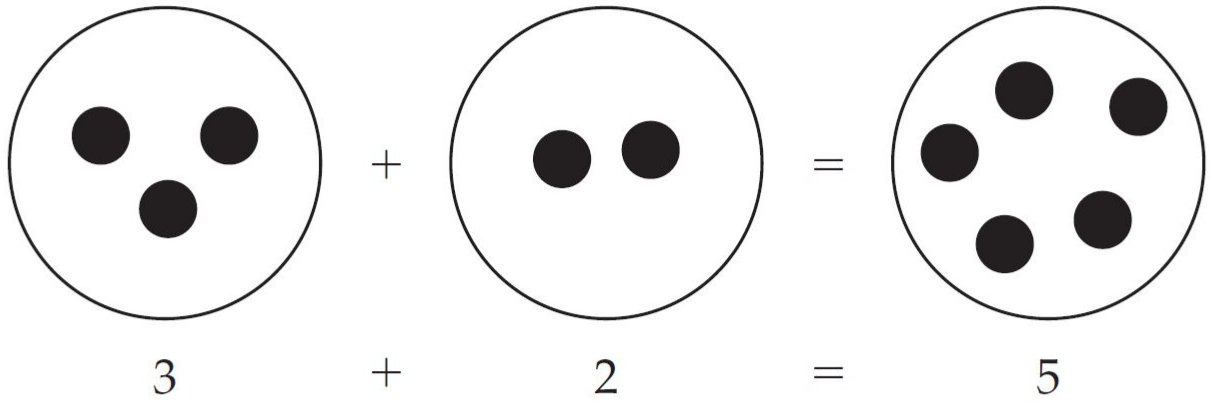
A Set Model for Integer Addition

We can use circles of different colors to model positive and negative numbers. Suppose we let red chips be negative numbers and black chips be positive numbers.

So, to add two positive numbers, for example 3 + 2, we would lay out a set of 3 black chips and a set of 2 black chips.

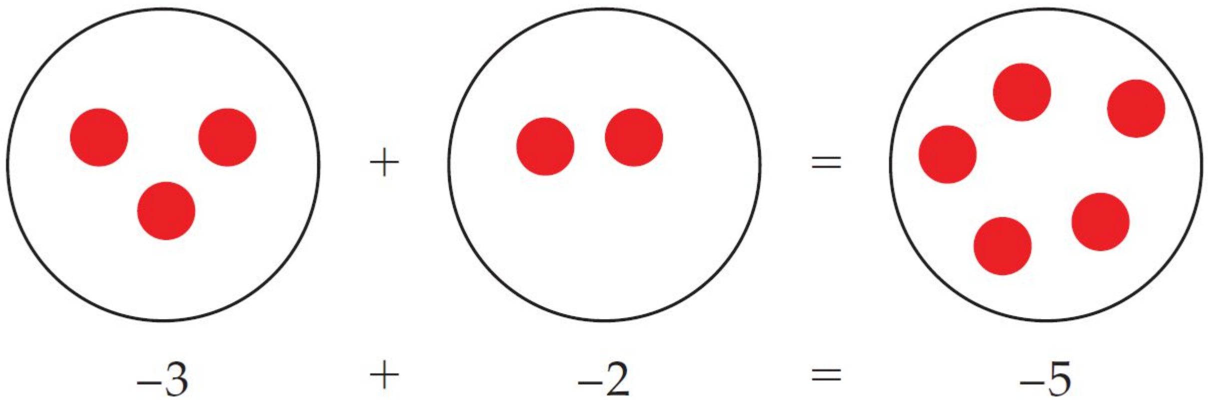
A Set Model for Integer Addition

Together this would give us 5 black chips, so the answer is +5.



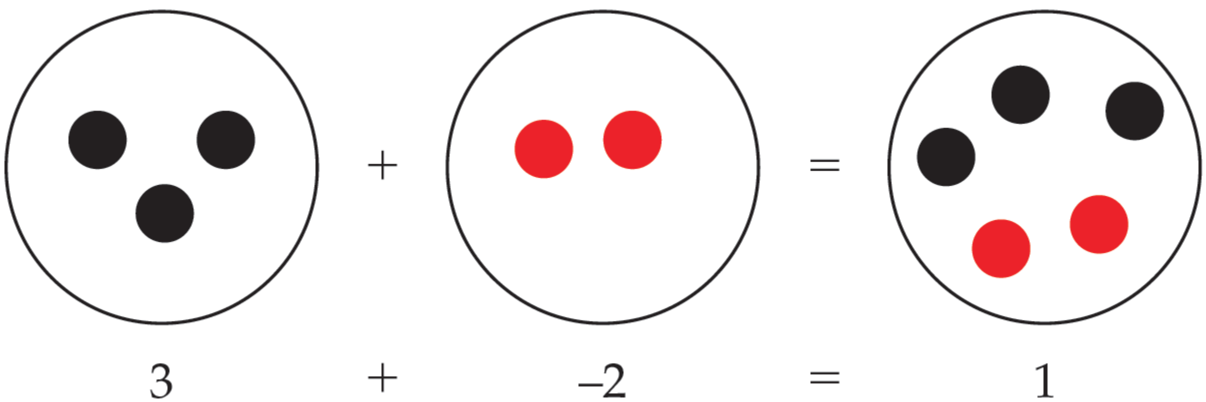
A Set Model for Integer Addition

Similarly, to add (−3) + (−2), put red chips instead of black to look like this:



A Set Model for Integer Addition

Now let's look at the cases where one number is negative and one is positive. For example, 3 + (−2) would look like this:



A Set Model for Integer Addition

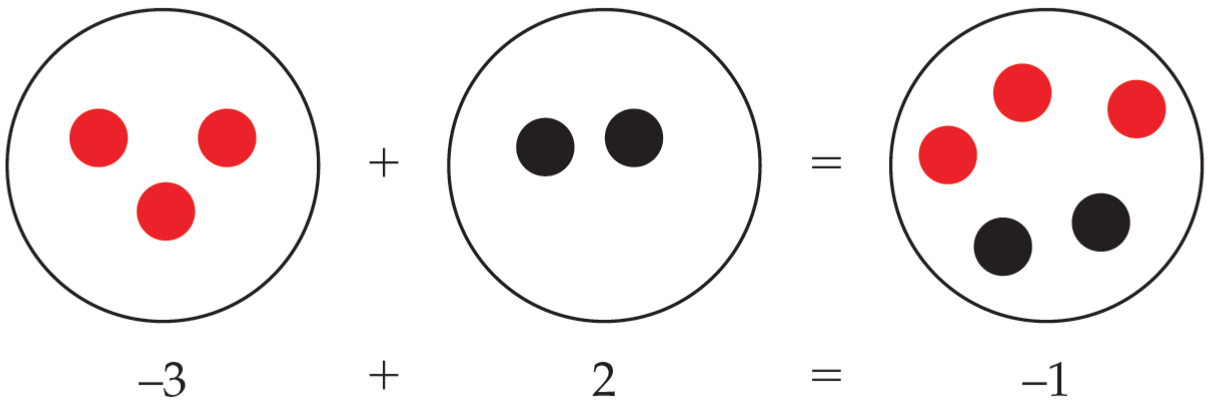
Here we have to consider what this answer is.

Wherever we have a pair of red and black, because this is $(-1) + 1$, this equals 0.

There is only 1 black chip left after taking out the pairs of 0, so the answer is +1.

A Set Model for Integer Addition

Similarly, if we do (−3) + 2, it would look like:



After taking out the pairs that equal 0, we end up with one red, so our answer is −1.

Understanding Subtraction with Integers

Understanding Subtraction with Integers

Let’s now focus on connecting whole-number subtraction to integer subtraction.

Investigation 3.4a – *Subtraction with Integers*

Do the following subtraction problems yourself before reading on. As you work, check to make sure that you are using your understanding of integers rather than just guessing.

A. $14 - (-25) =$

B. $-5 - 17 =$

C. $-6 - (-8) =$

D. $-12 - 5 =$

Investigation 3.4a – *Discussion*

We can define subtraction of negative numbers in terms of addition, which most people understand more easily.

To make the connection stronger, let us examine how we might use what we know about subtraction to determine the answer to $4 - 6$. In one sense, we cannot “take away” 6.

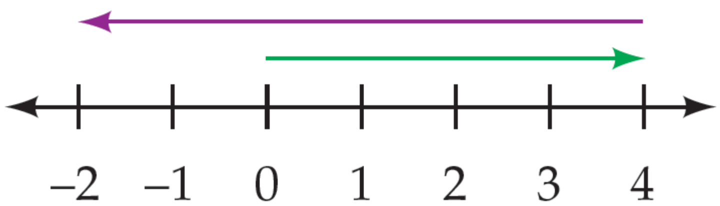
Investigation 3.4a – *Discussion* continued

However, if we think in terms of a checking account, if we take away 6 from 4, we will have a deficit of 2; that is, we have -2.

We could apply our work with subtraction and number lines: when we subtract one number from another, we move to the left.

Investigation 3.4a – *Discussion* continued

When we begin at 4 and move 6 units to the left, we end up at negative 2, as shown in Figure 3.18.



(Figure 3.18)

Thus, we can conclude that $4 + (-6) = -2$.

Investigation 3.4a – *Discussion* continued

We use this knowledge to define subtraction of integers formally in terms of addition:

$$a - b = a + (-b)$$

That is, subtracting is equivalent to adding the additive inverse.

Connecting Whole-Number Subtraction to Integer Subtraction

Connecting Whole-Number Subtraction to Integer Subtraction

In earlier section we have examined different models for addition and subtraction and different algorithms. In order for your knowledge of mathematics to be as connected as possible, it is important that you see how we apply these models to each new set of numbers.

Let us compare the two definitions:

$a - b = c$ if there is a number c such that $c + b = a$

$a - b = a + (-b)$

Connecting Whole-Number Subtraction to Integer Subtraction

If you examine the actual problems involved in subtracting a positive number from a positive number, you find that we are not adding the opposite as much as we are taking away a positive amount or comparing the size of two sets (each with positive values).

Therefore, although the definition given in this section seems simpler and more practical (to many students), it doesn’t connect as well to subtraction with two positive numbers.

Therefore, I chose to defer the definition until this section that the earlier work would be more focused on the context in which subtraction occurs when all three of the numbers (minuend, subtrahend, and difference) are positive numbers.

Connecting Subtraction Contexts to Algorithms

Connecting Subtraction Contexts to Algorithms

We now have developed efficient procedures for integer addition and subtraction.

However, as you have discovered, in many real-life settings, translating words into a mathematical sentence is not always simple.

Therefore, we will examine a few such settings to apply our knowledge.

Connecting Subtraction Contexts to Algorithms

Problem 1

Denine realized that she had overdrawn her checking account by \$60, and she was fined \$15 for a returned check. What is her present balance?

If we translate this problem into mathematical language, we find that we need to take away from negative \$60. Thus, the problem is $-60 - 15$. Using our understanding of subtraction, we translate this subtraction problem into the following addition problem: $-60 + (-15)$.

Connecting Subtraction Contexts to Algorithms

Applying our understanding of integer addition, we have an answer of −75; that is, her present balance is negative $75 (she is $75 in the red).

Problem 2

On one day the high temperature in Nome, Alaska, was −6 degrees. On that same day, the high temperature at the North Pole was −64 degrees. How much warmer was Nome than the North Pole?

Connecting Subtraction Contexts to Algorithms

If we translate this problem to mathematical language, we find that we are using the comparison model of subtraction. Thus, the problem is $-6 - (-64)$, which we can translate as $-6 + (+64)$, and the answer is 58; that is, it was 58 degrees warmer in Nome. A visual representation of this can help us make sense of it as well.