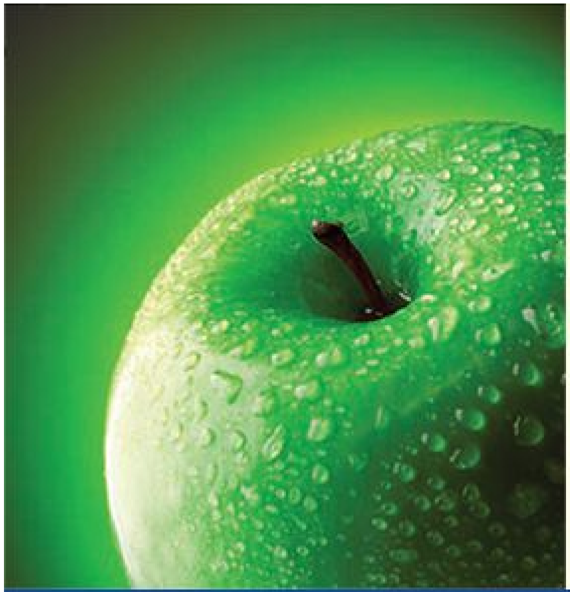


3 Understanding Addition and Subtraction

SECTION 3.1

Understanding Addition of Whole Numbers

What do you think?

- How many words, aside from *combine*, can you think of that describe addition?
- Not having place value, how might the Romans have added, for example, XXXVIII + XXVI?

Contexts for Addition

Contexts for Addition

1. Andy has 3 marbles, and his older sister Bella gives him 5 more. How many does he have now?

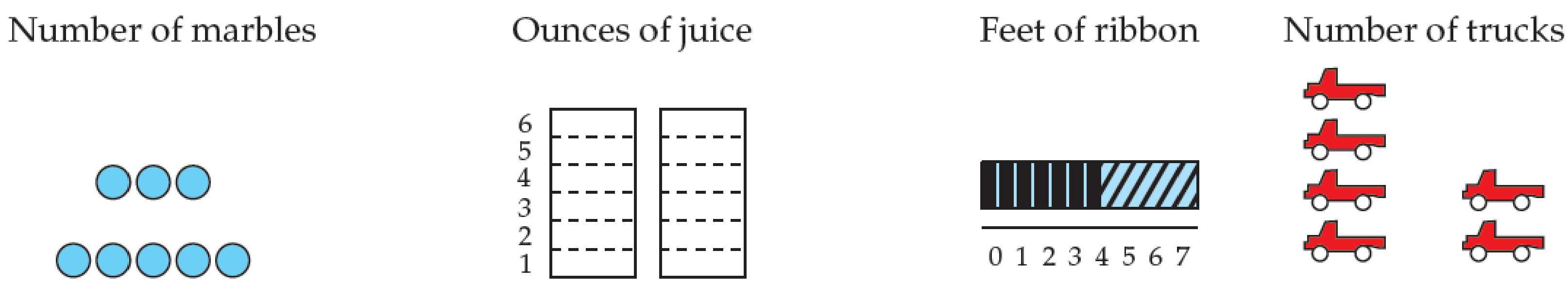
2. Keesha and José each drank 6 ounces of orange juice. How much juice did they drink in all?

3. Linnea has 4 feet of yellow ribbon and 3 feet of red ribbon. How many feet of ribbon does she have?

4. Josh has 4 red trucks and 2 blue trucks. How many trucks does he have altogether?

Contexts for Addition

How might a child that did not know addition solve each of these problems shown in Figure 3.1 using concrete tools?



Concrete Models for Addition

Figure 3.1

Problems 1 and 4 are easier for most children, because the child can see the actual marbles and trucks and count 1, 2, 3, 4, 5, 6, 7, 8 marbles and 1, 2, 3, 4, 5, 6 trucks.

Contexts for Addition

Problems 2 and 3 are more abstract in that the child cannot actually see 6 ounces or 4 feet. The child might represent the problem with concrete objects, such as 4 buttons for the yellow ribbon and 3 buttons for the red ribbon.

These problems represent two basic contexts in which we operate on numbers. Some numbers represent **discrete** amounts, or objects in a set, and some numbers represent **measured** or **continuous** amounts.

A Pictorial Model for Addition

A Pictorial Model for Addition

One of the themes of this book is the power of “multiple representations,” and each of the models used above seem on the surface to be quite different.

In each case, we are joining two sets or we are increasing a set. We can highlight the similarities among all addition contexts with the representations in Figure 3.2.

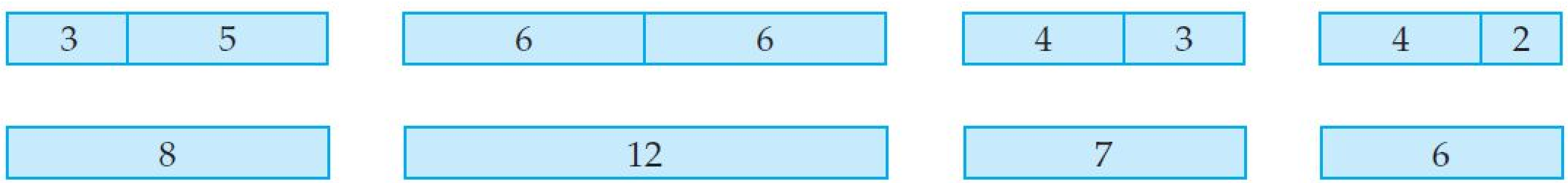


Figure 3.2

A Pictorial Model for Addition

In general, we can represent any addition $a + b = c$ as shown in Figure 3.3.

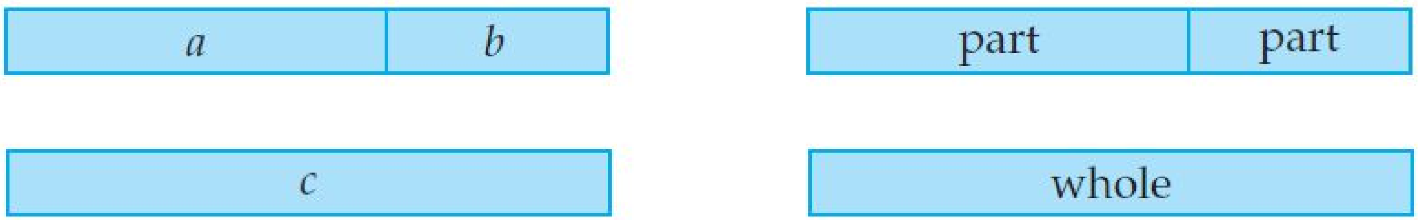


Figure 3.3

The numbers we are adding are called the **addends**, and the answer we get when we add is called the **sum**.

What advantages do you see for this general model over having no model at all?

A Pictorial Model for Addition

1. This model captures the way in which *all* addition problems are similar—that is, joining and combining two amounts to make a larger amount.
2. This model is also related to the notion of parts and wholes, an abstraction that is important in the development of whole-number ideas and in understanding other mathematical ideas, like fractions.
3. This model also works well whether the elements to be combined are sets of discrete objects, like the marbles and trucks, or measurements, like the ounces of juice or feet of ribbon.

A Pictorial Model for Addition

4. As we examine all four operations, we will see that we can define all four operations in this context. This reveals the essential connectedness of the four operations. When students see this connectedness, they are likely to be more successful with nonroutine and multistep problems. Students in Singapore, who consistently rank high in standardized math tests taken by students in over 100 countries, use similar bar models extensively to understand math. We will continue to explore these bar models in our discussion of fractions and algebra.

A Pictorial Model for Addition

It is important to note that understanding part–whole relationships with whole numbers allows numbers to be interpreted simultaneously as positions on the mental number line and as compositions of other numbers.

For example, 18 is the number after 17 and before 19, but 18 can also be seen as $10 + 8$, $9 + 9$, $20 - 2$, and so on.

A Pictorial Model for Addition

Understanding that a number can be composed (put together) and decomposed (broken into parts) is essential for being able to work confidently with the four operations.

This notion of composing and decomposing is one of the big ideas of elementary mathematics.

Representing Addition with Number Lines

Representing Addition with Number Lines

Number lines are found on rulers, clocks, graphs, and thermometers and can be a nice pictorial representation of the operations.

A **number line** can be constructed by taking a line (not necessarily a straight line) and marking off two points: zero (the origin) and one.

The distance from 0 to the point 1 is called the unit segment, and the distance between all consecutive whole numbers is the same.

Representing Addition with Number Lines

Although number lines are most commonly used to represent length, they may be used to model all kinds of problems.

For example, we could use a number line to indicate time, with each unit representing one unit of time—day, minute, year, and so on.

Moving from Concrete to Abstract

Moving from Concrete to Abstract

There are several classical stages in children’s understanding of addition. At the most basic level, the child counts to determine the sum.

For example, in the first problem, many young children would answer the question by putting the marbles on the floor and then counting the two groups.

In the next stage of development, the child “counts on.” That is, the child begins with the first number and counts however many more the second number represents.

Moving from Concrete to Abstract

A child solving Problem 1 (see Figure 3.4) in this manner would say, “4, 5, 6, 7, 8,” probably keeping track of the second number being added (5) with fingers.

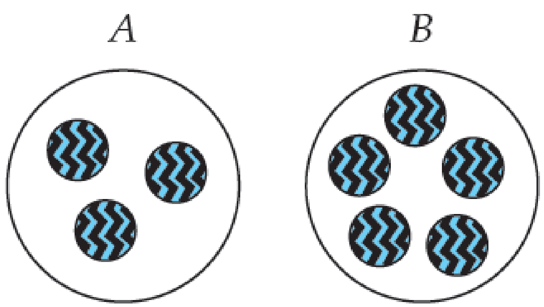


Figure 3.4

At the next level, the child realizes that it is possible to begin with the larger number (i.e., 3 + 5 = 5 + 3) and counts, “6, 7, 8.”

Moving from Concrete to Abstract

Finally, the child simply knows that 3 + 5 = 8.

In the diagram at the left in Figure 3.5, we first draw an arrow (in this case representing the length of the ribbon) 4 units long.

Figure 3.5

We draw another arrow 3 units long and connect the two arrows. The arrow at the top represents the combined length of the two shorter arrows.

Moving from Concrete to Abstract

In the diagram at the right, we start at the point on the line representing the length of the first ribbon and then draw an arrow 3 units long (representing the second ribbon). The location where the arrow ends tells us the combined length of the two ribbons.

Both diagrams represent 4 + 3 on the number line, although the one on the left more closely resembles the actual laying of the two ribbons end to end.

Investigation 3.1a – *Why Is the Sum of Two Even Numbers an Even Number?*

In the 2009 NCTM Yearbook entitled *Teaching and Learning Proof Across the Grades: A K −16 Perspective,* Deborah Schifter describes third-graders working on the question of how to prove that the sum of two even numbers is even.

Examine the following responses by the students and think about whether they constitute a proof:

> Paul: I know that the sum is even because my older sister told me it always happens that way.

Investigation 3.1a – *Why Is the Sum of Two Even Numbers an Even Number?* continued

Zoe: I know it will add to an even number because 4 + 4 = 8 and 8 + 8 = 16.

Evan: We really can't know! Because we might not know about an even number, and if we add it with 2, it might equal an odd number!

Melody: (Pointing to two sets of cubes she had arranged) This number is in pairs (pointing to the light-colored cubes), and this number is in pairs (pointing to the dark-colored cubes), and when you put them together, it's still in pairs.

Investigation 3.1a – *Discussion*

In what ways are these students communicating their understanding, building a logical progression of ideas, and using drawings to communicate their thinking? Shifter describes four categories of justification common to elementary students (and, I find, with college students too):

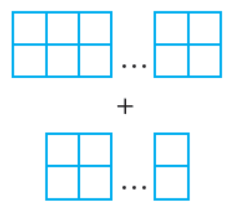
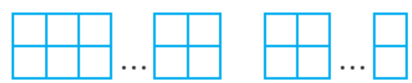
appeal to authority (Paul),

inference from instances (Zoe),

assertion that claims about an infinite class cannot be proven (Evan), and

reasoning from representation or context (Melody).

Do you see these categories in the students? In the diagrams below, you can see how Melody’s assertion closely parallels a more formal proof.

An integer is even if it can be represented as 2 times another integer.	A number is even if it can be broken up into pairs.
If a and b are even numbers, then we can find two integers x and y such that $a = 2x$ and $b = 2y$.	These two numbers are even because they can be broken into pairs. 
So $a + b = 2x + 2y$; but then $a + b = 2(x + y)$.	If you put them together, you still have pairs. 
Thus, $a + b$ is equal to 2 times an integer, but that is the definition of an even number.	Therefore, the sum of the two numbers is also even.

Investigation 3.1a – *Discussion* continued

Schifter asserts that young children are capable of making and justifying mathematical generalizations and that making arguments from representations (physical objects, pictures, diagrams, or story contexts) is an effective way to help students develop such reasoning capacity.

She proposed three criteria for such representations:

1. The meaning of the operation(s) involved is represented in diagrams, manipulatives, or story contexts.

Investigation 3.1a – *Discussion* continued

2. The representation can accommodate a class of examples.

3. The conclusion of the claim follows from the structure of the representation.

Do you see how Melody's argument satisfied these criteria?

1. Her representation modeled two whole numbers.

Investigation 3.1a – *Discussion* continued

2. Her language did not say 10 + 16 but rather two whole numbers. That is, her argument did not depend on the actual value of the two numbers (as Zoe's did).

3. When you place the two diagrams together, the resulting amount can also be represented in pairs.

Investigation 3.1b – *Precision with Definitions and Symbols*

A. Which of the following definitions for even numbers is more precise?

Definition one: An even number has 0, 2, 4, 6, or 8 in the ones place.

Definition two: When an even number is divided by 2, you get a whole number with none left over.

B. Research has shown that when asked to fill in the blank to the following, some students will insert a 5. Why do you think they do this?

$$3 + 2 = \_\_\_ + 1$$

Investigation 3.1b – *Discussion*

A. Although definition one does provide a way to recognize even numbers, it is not very precise for a couple of reasons. For one, a number like 2.4 would be even under the first definition as there is a 2 in the ones place.

However, 2.4 is not an even number. This definition also does not tell what an even number really is.

Definition two is more precise because it excludes numbers like 2.4 from meeting the definition and it explains precisely what an even number is.

Investigation 3.1b – *Discussion* continued

B. Some students have the misconception that an equals sign is a call to action to perform the operation, instead of realizing that the two sides of the equals sign have to be balanced.

These students will see the sign as a call to add 3 + 2 without paying attention to the 1. For now it is an illustration of how the equals sign may not be as intuitive as you previously believed.

Properties of Addition

Properties of Addition

When young children start to learn about adding, Table 3.1 has been the traditional method of representing the 100 "addition facts" that they have to learn.

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Table 3.1

Properties of Addition

It can look imposing to some children! However, understanding some properties of addition and base ten can unlock its potential as a learning tool.

As you look at the table, what do you observe (insights or patterns) that might make learning the addition facts easier for children?

One observation (in children's language) is that "adding zero doesn't change your answer."

Properties of Addition

Mathematicians call this the **identity property of addition**. It is represented in symbols as follows:

$$a + 0 = 0 + a = a$$

We also find that when we add any two numbers, we get the same sum regardless of the order in which we added, which is the **commutative property** of addition.

$$a + b = b + a$$

Properties of Addition

There is another discovery, called “bridging with 10,” that makes learning addition easier.

For example, if you ask a child, “What is $7 + 5$?” many children will say something like, “$7 + 3$ is 10 and 2 more is 12.” Many will intuitively decompose $7 + 5$ into $7 + 3 + 2$.

If we write this as $7 + (3 + 2) = (7 + 3) + 2$, then this leads to a third property, called the **associative property of addition**. As another example,

$$(37 + 75) + 25 = 37 + (75 + 25).$$

Properties of Addition

Formally, we say

$$(a + b) + c = a + (b + c)$$

One last property of addition often seems almost trivially obvious; it is called the **closure property of addition**.

The closure property states that the sum of any two whole numbers is a unique whole number. There are two parts to this property: (1) uniqueness (the sum will always be the same number) and (2) existence (the sum will always be a whole number).

Properties of Addition

Not all sets are closed under addition, for example, the set of odd whole numbers.

The sum of two odd numbers is not in the set of odd numbers because an odd plus an odd equals an even; thus, we say that the set of odd numbers is not closed under addition.

Investigation 3.1c – *A Pattern in the Addition Table*

Patterns can help make the learning of mathematics easier and more interesting. In the addition table, if you look at any 2 × 2 **matrix** (that is, a rectangular array of numbers or other symbols), the sums of the numbers in each of the two diagonals are equal.

6	7
7	8

For example, in the matrix to the left, 6 + 8 = 7 + 7. Can you justify this pattern mathematically?

Investigation 3.1c – *Discussion*

Description 1

Verbally, you can justify this pattern by saying that in any 2×2 matrix in the table, the two numbers in one diagonal are always identical and the other two numbers are always 1 less and 1 more than this number. Therefore, the sum of the two other numbers will “cancel out” so that you get the “same” sum in either case.

Description 2

We can use some notation to make the description simpler.

Investigation 3.1c – *Discussion* continued

Noting that the value of each number increases by 1 each time that we move across (or down) the table, we can let *x* represent the number in the top left corner of the diagonal.

Thus, in relation to *x*, the values of the other three numbers are

x	$x+1$
$x+1$	$x+2$

Investigation 3.1c – *Discussion* continued

It is an algebraic exercise to demonstrate that the sum of each diagonal is $2x + 2$. Going from top left to bottom right, we have $(x) + (x + 2) = 2x + 2$. The other diagonal is $(x + 1) + (x + 1)$, which also equals $2x + 2$.

Description 3

Yet other students will say that the sums of the diagonals are equal because “it’s the same numbers in both cases.” What do you think such a student might be seeing? Think before reading on. . . .

Investigation 3.1c – *Discussion* continued

Let’s say that we are looking at the matrix formed by the intersection of the 2 row and the 3 row and the 4 column and the 5 column.

The numbers are 6, 7 and 7, 8. However, if we represent the numbers by their origin, we have

	4	5
2…	$2 + 4$	$2 + 5$
3…	$3 + 4$	$3 + 5$

Investigation 3.1c – *Discussion* continued

The sum of the top-left-to-bottom-right diagonal is (2 + 4) + (3 + 5). However, because of the commutative and associative properties, this sum is equal to (2 + 5) + (3 + 4). In other words, we are indeed using the same numbers!

We can now generalize this cell by saying that the matrix formed by the intersection of the *a* row and the *b* row and the *c* column and the *d* column is

$$\begin{array}{r|c|c|} & c & d \\ \hline a\ldots & a+c & a+d \\ \hline b\ldots & b+c & b+d \\ \hline \end{array}$$

and $a + c + b + d = a + d + b + c$

Investigation 3.1c – *Discussion* continued

Our first work with addition in base ten will be doing some addition problems mentally, for a couple of reasons. First, this will require you to think carefully about how your knowledge of place value applies to adding numbers.

Second, much of our use of arithmetic does not involve pencil and paper or calculators, but rather mental computation—when estimating or when it is quicker to do a computation or part of a computation in our head than with a pencil or calculators.

Trading versus Carrying and Borrowing

Trading versus Carrying and Borrowing

Most of us learned to use the words "carry" when we add and "borrow" when we subtract. We need to change that language to be more descriptive and to show the similarities between what is happening in each case.

The following problems illustrate this point.

$$\begin{array}{r} {}^{1} \\ 36 \\ +28 \\ \hline 64 \end{array} \qquad\qquad \begin{array}{r} {}^{5}\phantom{{}^{1}4} \\ \cancel{6}\,{}^{1}4 \\ -3\;\;6 \\ \hline 2\;\;8 \end{array}$$

Trading versus Carrying and Borrowing

In the addition problem, because 6 + 8 > 9, we put the 4 in the ones place and “carry” the 1 to the tens place.

In the subtraction problem, in order to subtract in each place, we need to “borrow” a 1 from the tens place and move it to the ones place whose value is now 14.

Take a minute to write down what is similar or the same about both processes and what is different. Also, consider what “carrying” and “borrowing” mean in other contexts and whether they are really descriptive of this process.

Trading versus Carrying and Borrowing

What is the same about both is that (1) an amount is moved from one place to another, and (2) this amount always represents a 10-for-1 exchange (10 ones for 1 ten or 1 ten for 10 ones).

What is different is the direction of the exchange. When adding, we move the 10 from the smaller place to a 1 in the larger place (right to left). When subtracting, we move the 1 from the larger place to a 10 in the smaller place.

Investigation 3.1d – *Mental Addition*

Do the following five computations in your head. Briefly note the strategies you used, and try to give names to them.

Note: One mental tool all students have is being able to visualize the standard algorithm in their heads. For example, for the first problem, you could say: “9 + 7 = 16, trade for ten, then 5 + 3 = 8 and 8 + 1 traded ten makes 9; the answer is 96.”

Investigation 3.1d – *Mental Addition* continued

However, because you already know that method, I ask you not to use it here but to try others.

There are actually quicker ways to do this problem in your head than using the traditional algorithm. See whether you can discover any of them.

1. 39 + 57 **2.** 68 + 35 **3.** 66 + 19 **4.** 545 + 228
5. 186 + 125

Investigation 3.1d – *Discussion*

Leading digit

One strategy that works nicely with most addition problems is **leading digit**.

Some people refer to it as front end because we add the "front" of the numbers first. Using leading digit with Problem 1 looks like this:

$$39 + 57 = (30 + 50) + (9 + 7) = 80 + 16 = 96.$$

Investigation 3.1d – *Discussion* continued

This strategy can be used with larger numbers too.

545 + 228 = (500 + 200) + (40 + 20) + (5 + 8)= 700 + 60 + 13 = 773

Compensation

Another powerful mental math strategy is called **compensation**. Using compensation with Problem 1 looks like this: 39 + 57 = 40 + 56.

Investigation 3.1d – *Discussion* continued

Do you see how we transformed 39 + 57 into 40 + 56?

Which other problems lend themselves to this strategy?

Number 3 could also be solved with this strategy:

$$66 + 19 = 65 + 20$$

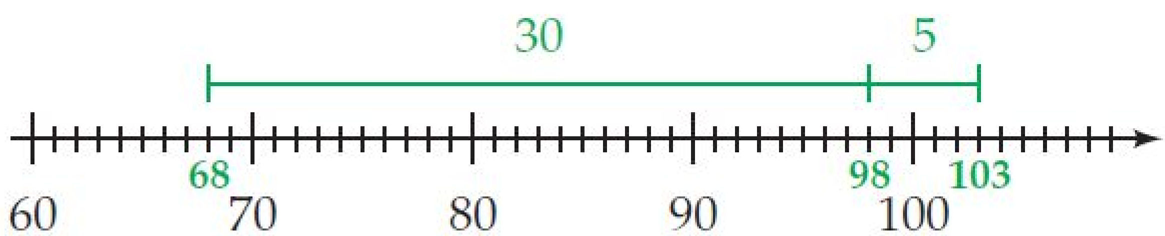
Break and bridge

We can use the break and bridge strategy in Problem 2 in this way:

$$68 + 35 = (68 + 30) + 5 = 98 + 5 = 103$$

Investigation 3.1d – *Discussion* continued

Representing this strategy on a number line makes it easier for some to understand. We break the second number apart (using expanded form) and add one place at a time.



Which other problems lend themselves to this strategy?

Investigation 3.1d – *Discussion* continued

Compatible numbers

Another powerful strategy is creating compatible numbers. This often involves seeing pairs of digits whose sum is 10.

Using compatible numbers with Problem 5 looks like this:

$$186 + 125 = (180 + 120) + (6 + 5) = 300 + 11 = 311$$

This works when we see that 180 + 120 = 300.

Investigation 3.1d – *Discussion* continued

Choosing a strategy

Which strategy you use is often a matter of preference. For example, Problem 2 (68 + 35) may be done mentally in at least four different ways, each of which is the easiest way for *some* students.

One of your goals as a future teacher is to become comfortable with each strategy so that you can support learning for all of your students.

Investigation 3.1d – *Discussion* continued

Leading digit: 68 + 35 = 60 + 30 + 8 + 5

Compensation: 68 + 35 = 70 + 33

Break and bridge: 68 + 35 = 68 + 30 + 5

Compatible numbers: 68 + 35 = 65 + 35 + 3

Justifying strategies

Let us look at the compatible numbers strategy for Problem 2 in detail to connect the mental work with the properties we have discussed.

Investigation 3.1d – *Discussion* continued

The action	Justification
$68 + 35 = (65 + 3) + 35$	Substitution
$= 65 + (3 + 35)$	Associative
$= 65 + (35 + 3)$	Commutative
$= (65 + 35) + 3$	Associative
$= 100 + 3$	Addition
$= 103$	Addition

Children’s Strategies for Addition

Children's Strategies for Addition

Here we will focus on the mathematics underlying common stages in the child's development of computation with addition. Imagine that you are a child and you haven't yet learned the standard procedure for adding. How might you add 48 + 26?

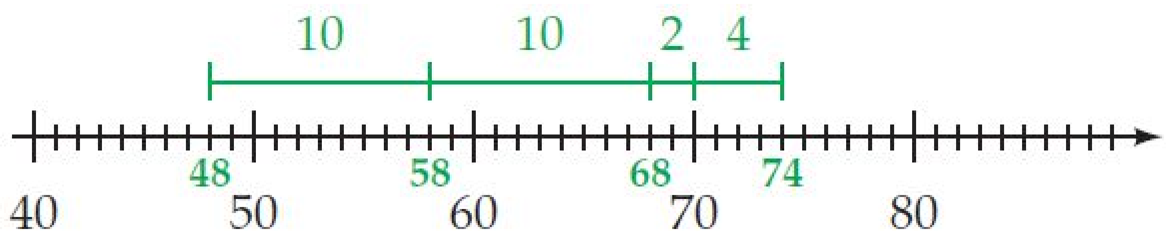
Add up by 10s

Just as you read earlier that one stage in young children's development of addition is to add up, some children begin multidigit addition in the same way.

Children's Strategies for Addition

They add up by 10s from 48. What they say is "48, 58, 68, 74." Some children will struggle a bit from 68 to 74.

They use 10 as a bridge, and say, "48, 58, 68, plus 2 is 70, plus 4 more is 74." Many children will use a number line to explain this strategy.



Children's Strategies for Addition

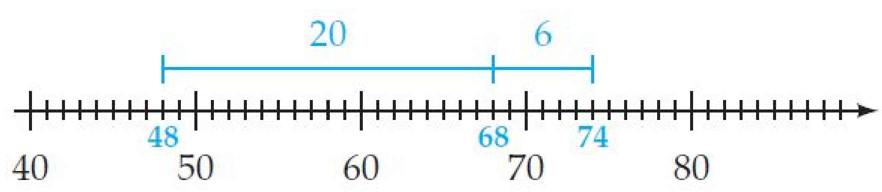
Break the Second Number Apart

Another variation of adding up is to begin with the first number and break the second number into its place value parts. That is, they add the tens and then the ones.

$$48 + 20 = 68$$

$$68 + \ \ 6 = 74$$

A number line is a useful representation of this strategy.

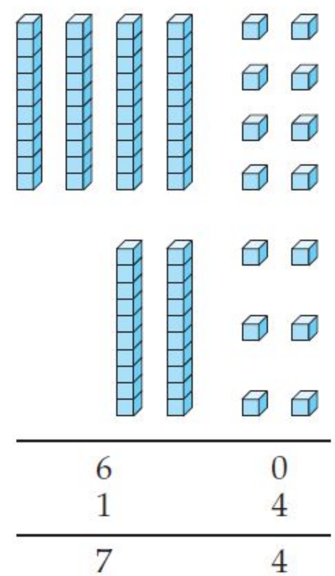


Children's Strategies for Addition

Use Partial Sums

A very common approach that comes closer to one of the standard algorithms is to add, from left to right. In the beginning, children often write the **partial sums** (which are the sums of each place). The diagram below illustrates this approach with manipulatives.

$$
\begin{array}{r}
48 \\
+\ 26 \\
\hline
60 \\
14 \\
\hline
74
\end{array}
$$



Children's Strategies for Addition

Note that this method is essentially the leading-digit method that many people use when adding mentally.

Use Money and Compensate

Depending on the problem, some children will use other frameworks, often money.

For example, because 48 and 26 are close to a half-dollar and a quarter, some children will add 50 and 25 and then compensate, since 48 is 2 less than 50 and 26 is 1 more than 25.

Children’s Strategies for Addition

$$50 + 25 - 2 + 1$$

This strategy, though it is not applying base ten in this problem, is actually quite powerful and can be connected to base ten through discussion.

Adding Left to Right

Constance Kamii and many others who have explored what is called a constructivist approach to learning arithmetic have found that when children are not just shown how to add, the vast majority of children will actually add from the largest to the smallest place.

Children's Strategies for Addition

This is very interesting because we read from left to right. So, to add 48 and 26 this way, first add 40 + 20 = 60, then 8 + 6 = 14, so the total is 60 + 14 = 74.

Once you get the hang of this strategy, it can be much faster than other methods.

Investigation 3.1e – *Children’s Strategies for Adding Large Numbers*

What if the numbers were bigger—for example, 368 + 574? Look back on the approaches described earlier.

Can you adapt any of them to this problem?

Investigation 3.1e – *Discussion*

The “break the second number apart” strategy applies to larger numbers.

$$
\begin{aligned}
368 + 500 &= 868 \\
868 + 70 &= 938 \\
938 + 4 &= 942
\end{aligned}
$$

The “partial sums” strategy also applies.

Investigation 3.1e – *Discussion* continued

$$300 + 500 = 800$$

$$60 + 70 = 130$$

$$8 + 4 = 12$$

$$800 + 130 = 930 \quad \text{and} \quad 930 + 12 = 942$$

The “partial sums” strategy can be modified to “keep each sum in its proper place,” which is shown below.

$$\begin{array}{r} 368 \\ +\,574 \\ \hline 800 \\ 130 \\ 12 \\ \hline 942 \end{array} \qquad \begin{array}{r} 368 \\ +\,574 \\ \hline 8 \\ 13 \\ 12 \\ \hline 942 \end{array} \qquad \text{or} \qquad \begin{array}{r} 368 \\ +\,574 \\ \hline 12 \\ 13 \\ 8 \\ \hline 942 \end{array}$$

Children's Strategies for Addition

Algorithms

A major goal of elementary school is to have students become computationally fluent. This means developing efficient algorithms for each operation.

An **algorithm** is a single, clearly described method that works in all cases.

The algorithms that you learned to add, subtract, multiply, and divide are not *the* algorithms but simply four of many.

Children's Strategies for Addition

Furthermore, they are not universally used today.

Some of you, in different parts of the country, learned different algorithms, and school children in different parts of the world learn very different algorithms for some of the operations, especially subtraction.

So let us examine this notion of algorithm.

Adding Before Base Ten

Adding Before Base Ten

Before we begin examining addition in base ten, take a moment to think about how people added before base ten was invented. Imagine that you were a Roman.

How might you add these two numbers? Romans couldn't just add the VII to the VI and "carry" the III, since it is not a place value system.

They would have to combine the symbols and do some rewriting where efficient (such as V + V = X).

Investigating Addition Algorithms in Base Ten

Investigating Addition Algorithms in Base Ten

The National Assessment of Education Progress (NAEP) has found that third-graders' success in computation decreases considerably when they move from two-digit problems to three-digit problems.

Further research indicates that many students are memorizing rather than understanding the process. Therefore, it is really important that you learn to truly understand the processes, rather than memorizing algorithms.

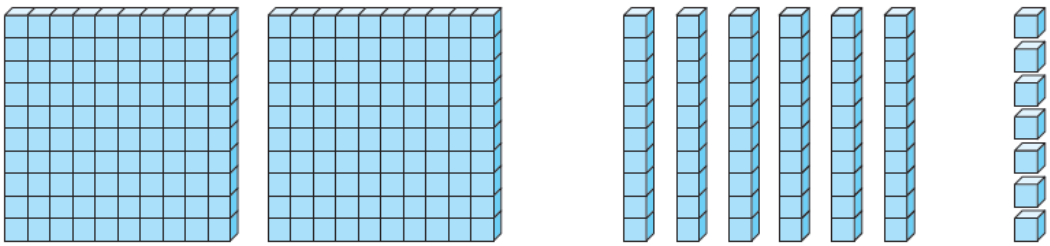
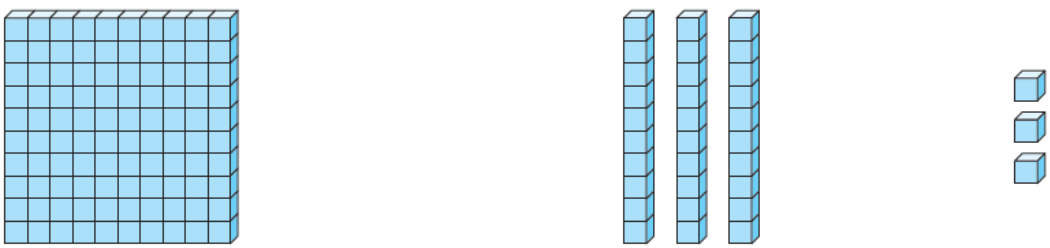
Investigating Addition Algorithms in Base Ten

Add 267 + 133 as you normally would. Can you explain the "whys" of each step?

In the context of our base ten numeration system, we are combining 2 hundreds, 6 tens, and 7 ones with 1 hundred, 3 tens, and 3 ones.

If the student understands the process of combining and regrouping, then this problem is *not* substantially more difficult than one with smaller numbers; it is only longer.

Investigating Addition Algorithms in Base Ten

Using manipulatives	Using words
	2 hundreds, 6 tens, 7 ones
	+
	1 hundred, 3 tens, 3 ones

Investigating Addition Algorithms in Base Ten

A commonly used algorithm

With this manipulative representation, you can now better understand the why of one **common algorithm for addition**.

Step 1

7 + 3 = 10; place the 0 in the ones place and put the 1 above the tens place, because 7 + 3 is equivalent to 1 ten and 0 ones.

$$\begin{array}{r} {}^{1} \\ 267 \\ \underline{133} \\ 0 \end{array}$$

Investigating Addition Algorithms in Base Ten

Step 2

1 + 6 + 3 = 10 (which is really 1 tens plus 6 tens plus 3 tens = 10 tens or 1 hundred); place the 0 in the tens place and put the 1 above the hundreds place (to represent trading 10 tens for 1 hundred).

$$\begin{array}{r} \scriptstyle{1\,1} \\ 267 \\ 133 \\ \hline 00 \end{array}$$

Step 3

1 + 2 + 1 = 4 (which is really 1 hundred plus 2 hundreds plus 1 hundred = 4 hundreds); place the 4 in the hundreds place.

Investigating Addition Algorithms in Base Ten

The sum is 400.

$$\begin{array}{r} {}^{1\,1} \\ 267 \\ \underline{133} \\ 400 \end{array}$$

Representing the problem in expanded form enables us to prove why it works.

Statement	Justification
$267 + 133$	
$= (2 \cdot 100 + 6 \cdot 10 + 7) + (1 \cdot 100 + 3 \cdot 10 + 3)$	Expanded form
$= (2 \cdot 100 + 1 \cdot 100) + (6 \cdot 10 + 3 \cdot 10) + (7 + 3)$	Commutative and associative properties
$= (2 + 1)100 + (6 + 3)10 + (7 + 3)$	Distributive property
$= 3(100) + 9(10) + 10$	Addition
$= 3(100) + (9 + 1)10$	Distributive property
$= 3(100) + 10(10)$	Addition
$= 3(100) + 100$	Multiplication
$= (3 + 1)100$	Distributive property
$= 4(100)$	Addition
$= 400$	Multiplication

Investigating Addition Algorithms in Base Ten

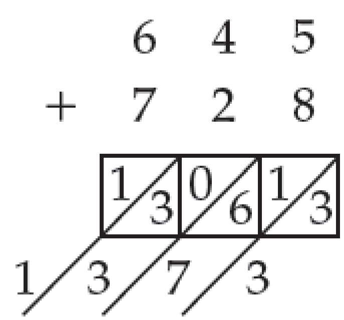
Seeing how numbers can be *composed* and *decomposed* makes it possible to understand the algorithm deeply.

To understand the addition algorithm, we **decompose** the number using expanded form; we can then see how those different parts can be reconfigured—**composed**—to make our new whole, that is, the sum.

Investigation 3.1f – *An Alternative Algorithm*

As mentioned earlier, when base ten was invented, many different algorithms for each operation were invented. One algorithm that is a favorite of children is called the **lattice algorithm**.

Observe the example below and see if you can figure out how it works and why it works.



Investigation 3.1f – *Discussion*

First, you write the problem, and below each place draw a square.

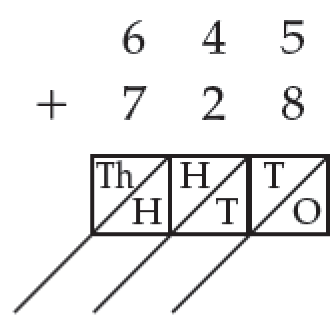
Second, draw diagonal lines through each square that extend below the square.

Third, write the result of each partial sum in the box.

Last, add diagonally. If the sum in any diagonal addition is greater than 10, trade to the next diagonal just as you do with the standard algorithm.

Investigation 3.1f – *Discussion* continued

If we represent the problem in terms of the place value of each part of the lattice, we see that the lattice "herds" each digit to the proper place.



O = Ones
T = Tens
H = Hundreds
Th = Thousands

Investigation 3.1g – *Addition in Base Five*

Do these addition problems in base five. Concrete models or pictorial representations are encouraged to help you understand.

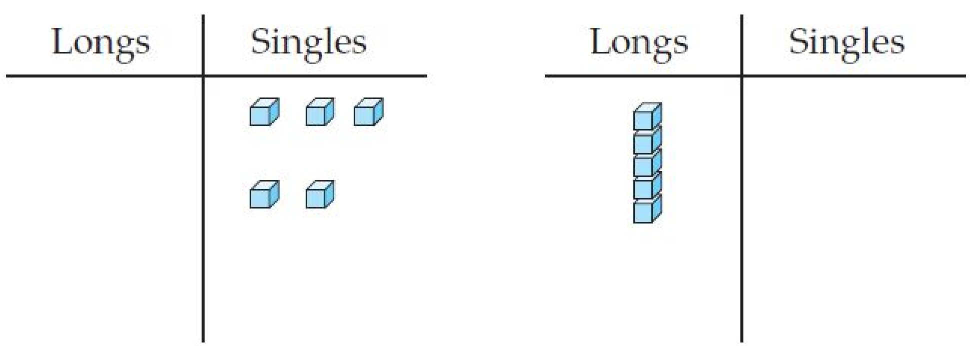
A. $3_{five} + 2_{five}$ **B.** $21_{five} + 13_{five}$ **C.** $43_{five} + 24_{five}$

Investigation 3.1g – *Discussion*

A. $3_{five} + 2_{five}$

One way to determine the sum, especially if this feels awkward, is to count on from 3 just like children do when learning to add in base ten.

When we do this, we get 3_{five}, 4_{five}, 10_{five}. If we represent this problem with manipulatives, we have



Investigation 3.1g – *Discussion* continued

Because in base five, we trade to the next place value when we have five in a place, the five singles become one long with zero singles left over. So,

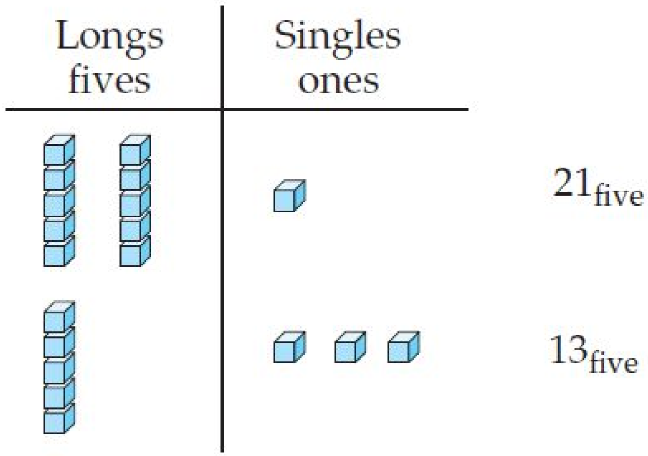
$$3_{\text{five}} + 2_{\text{five}} = 10_{\text{five}}.$$

B. $21_{\text{five}} + 13_{\text{five}}$

You might prefer to represent this problem vertically. Since there is no trading, the answer is 34_{five}.

$$\begin{array}{r} 21_{\text{five}} \\ +\ 13_{\text{five}} \\ \hline 34_{\text{five}} \end{array}$$

Investigation 3.1g – *Discussion* continued

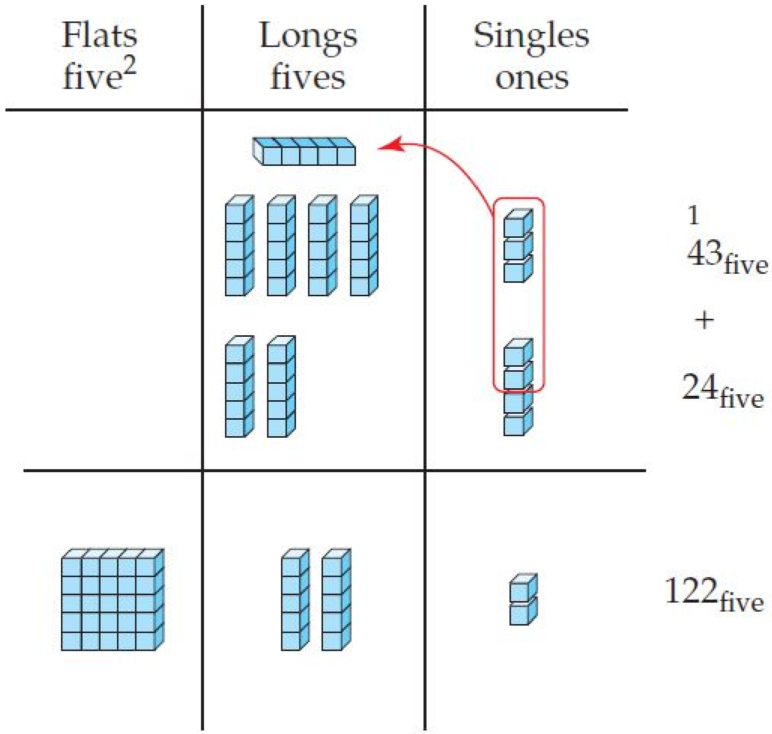


C. $43_{five} + 24_{five}$

Since there is trading here, we will solve the problem simultaneously with a concrete and pictorial representation.

Investigation 3.1g – *Discussion* continued

It is important to see the connections between these representations.



Investigation 3.1g – *Discussion* continued

We have to do some trading in this example. In the ones place, we trade five singles for 1 long of five and are left with 2 singles in the ones place.

Then we add the 4 longs + 3 longs + 1 long (the one long from the trading), and because we have more than five longs, we trade five longs for 1 flat and are left with 2 longs.

This gives us 1 flat and 2 longs and 2 singles; in other words, 122_{five}.

Investigation 3.1g – *Discussion* continued

While you will likely not teach adding in other bases in elementary school, this helps us to understand adding in base ten.

The process is the same—the only difference is how many we need in one place in order to trade for the next place.

Investigation 3.1h – *Children's Mistakes*

The problem below illustrates a common mistake made by many children as they learn to add.

Understanding how a child might make that mistake and then going back to look at what lack of knowledge of place value, of the operation, or of properties of that operation contributed to this mistake is useful.

What error on the part of the child might have resulted in this wrong answer?

The problem: 38 + 4 = 78

Investigation 3.1h – *Discussion*

In this case, it is likely that the child lined up the numbers incorrectly:

$$\begin{array}{r} 4 \\ +\ 38 \\ \hline 78 \end{array}$$

Giving other problems where the addends do not all have the same number of places will almost surely result in the wrong answer. For example, given 45 + 3, this child would likely get the answer 75.

Investigation 3.1h – *Discussion* continued

Given 234 + 42, the child would likely get 654. In this case, the child has not “owned” the notion of place value. Probably, part of the difficulty is not knowing expanded form (for example, that 38 means 30 + 8—that is, 3 tens and 8 ones).

An important concept here is that we need to add ones to ones, tens to tens, and so on. Base ten blocks provide an excellent visual for this concept as students can literally see why they cannot add 4 ones to 3 tens.

Estimation

Estimation

Most people now rely on calculators when exact answers are needed. However, estimating skills are still very important in cases where an exact answer is not needed and to check the reasonableness of results obtained on the calculator.

Estimation, in turn, requires good mental arithmetic skills, which come from an understanding of the nature of the operations, a firm understanding of place value, and the ability to use various properties.

Estimation

When are numbers estimates and not exact numbers?

Before we examine some methods of estimation, we need to understand when numbers represent estimates and when they represent exact amounts.

All of the following numbers are estimates or approximations. What ways can you see to group them according to why they are estimates?

For example, the age of a dinosaur bone is an estimate because present dating methods do not enable us to get an exact number; in other words, the exact age is unknown.

Estimation

- A certain dinosaur bone is 65 million years old.
- The number of hungry children in the United States is 16 million.
- The area of the Sahara Desert is 3,320,000 square miles.
- The mean July temperature in Tucson, Arizona, is 86 degrees.
- Jane lives 55 miles from the nearest airport.
- My office is 12 feet by 9 feet.

Estimation

Numbers are estimates when:

1. The exact value is unknown—for example, predictions and numbers that are too large or difficult to determine.

2. The value is not constant—for example, population and barometric air pressure.

3. There are limitations in measurement—for example, when we use measuring tools, there is a limitation to how precise we can be.

Estimation

Rounding

Just as many numbers in everyday life are estimates, many numbers are rounded:

- It took Jackie 10 hours to get from Boston to Buffalo.
- Anna gets 34 miles per gallon with her new car.
- Rosie put 2100 miles on her car last month.
- Fred paid $18,000 for his new car.
- The population of Sacramento, California, is 370,000.

Estimation

The preceding examples bring another question to mind: Why do we round?

1. Rounding makes comprehension easier.

2. Rounding makes computation easier.

When do we use estimation and when do we use exact computation?

Following are several examples of when people generally estimate:

- Making a budget—cost of college, cost of food per month

Estimation

- Determining the cost of a trip or vacation—ski trip, camping trip, trip to Europe
- Deciding which to buy—a new car or a used car
- Determining time—how long to get to . . .
- Determining whether we have enough money—being at the grocery store when short on cash
- Deciding how much the tip should be (at a restaurant)
- Determining how long the paper or project will take

Estimation

Estimation methods

As you will find in this chapter, we do not estimate in the same way in all situations. The method(s) we use to estimate depend partly on how close the estimate has to be and on whether we want to over- or underestimate deliberately.

Estimation Strategies for Addition

Estimation Strategies for Addition

Here we will analyze some estimation problems to understand better the application of base ten concepts and mental math strategies.

Investigation 3.1i – *What Was the Total Attendance?*

A. Approximately what was the total attendance for the following three football games at Tiger Stadium?

75,145 34,135 55,124

Investigation 3.1i – *Discussion*

Remember that one of the main goals of the investigations is for you to develop a repertoire of strategies.

Leading digit:	Add the “leading digits”
	7 + 3 + 5 = 15; that is, 150,000
	The leading-digit method used alone will always give you an estimate that is lower than the actual sum.

Investigation 3.1i – *Discussion* continued

Refined leading digit: Add the leading digits as above, then also add the next digits, and then add these together.

150,000 + (5 + 4 + 5 = 14—that is, 14,000)

= 150,000 + 14,000 = 164,000

Rounding: Round to the nearest ten thousand.

80,000 + 30,000 + 60,000
= 170,000

Investigation 3.1i – *Discussion*

continued

Compatible numbers: Round to numbers that are "compatible" or easy to add.

$75 + 35 + 55 = 110 + 55 = 165$, which represents 165,000

Investigation 3.1i – *What Was the Total Attendance?*

B. Approximately what was the total attendance for the following three baseball games at Wrigley Field in Chicago?

32,425 31,456 34,234

Investigation 3.1i – *Discussion*

Each of the four methods above could have been used here. You may well have come up with another strategy called **clustering**, because all three numbers are relatively close together.

In this case, a very quick, rough estimate would be

$$30{,}000 \times 3 = 90{,}000$$

If we use a refined leading-digit strategy, we can get 90,000 + 7000, and looking at the 425, 456, and 234, we can see that this is about 1000.

Investigation 3.1i – *Discussion* continued

Thus, a more refined estimate is about 98,000.

Looking back

There are two points to keep in mind when estimating and doing mental mathematics:

1. The method you use is often partially determined by the problem itself. If you want only a rough estimate, you might use leading digit or rounding. If you want a more refined estimate, you might use compatible numbers.

Investigation 3.1i – *Discussion* continued

If you want to make sure you have enough money, you might round everything up so that the estimated sum is definitely greater than the actual sum.

2. If you have a large repertoire of estimating and mental math techniques in your toolbox, you will be more skillful, and in a class of 20 children, you will see many different strategies.

Investigation 3.1j – *Estimating by Making Compatible Numbers*

Using compatible numbers is an effective strategy to estimate the following sums. Try this on your own and then check below.

A.

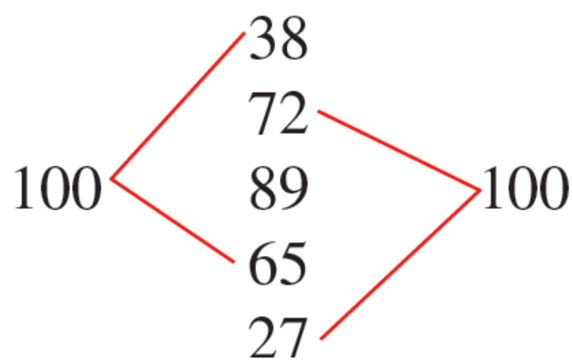
$$\begin{array}{r} 38 \\ 72 \\ 89 \\ 65 \\ +\ 27 \\ \hline \end{array}$$

B.

$$\begin{array}{r} 23 \\ 359 \\ 177 \\ 675 \\ 162 \\ +315 \\ \hline \end{array}$$

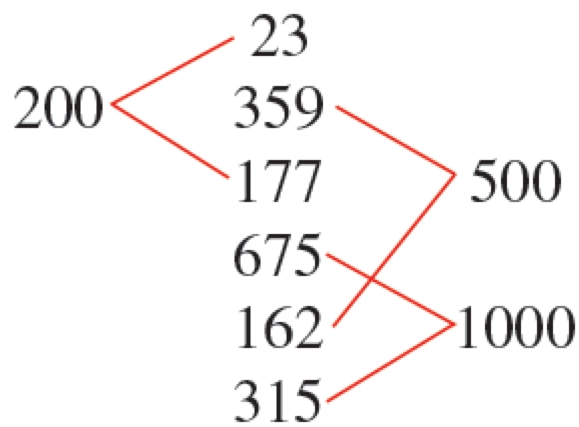
Investigation 3.1j – *Discussion*

A. In this case, a quick glance shows us that 38 and 65 will make a sum close to 100, and so will 72 + 27. If we see this, our estimate of 200 + 89 = 289 is quite close to the actual answer of 291.



Investigation 3.1j – *Discussion* continued

B. Using compatible numbers, we can estimate
200 + 500 + 1000 = 1700.



Number Sense

Number Sense

Number sense involves the ability:

1. To take numbers apart and put them together.

2. To move fluently among different representations.

3. To recognize when one representation is more useful than another.

4. To perform mental computation and estimation flexibly.

5. To determine whether an answer is reasonable.

Number Sense

In each section of this chapter, we will investigate problems that will help to further develop your number sense. It is important to note that number sense is being developed throughout; however, we will use some problems to focus explicitly on number sense.

Investigation 3.1k – *Number Sense with Addition*

A. I'm thinking of 3 numbers whose sum is greater than 70.

Tell whether each of the following must be true, might be true, can't be true:

1. All three numbers are greater than 20.

2. If two of the numbers are less than 20, the other must be greater than 20.

There are generally two ways to proceed. One is to try to explain why it *must* be true. Another is to find a counterexample to demonstrate that the statement is not true.

Investigation 3.1k – *Discussion*

1. Here is a counterexample: 10 + 30 + 40 = 80. The sum is greater than 70, but only two of the three numbers are greater than 20. So, this statement might be true (as in the case of 25 + 30 + 30), but it is not always true as this counterexample proves.

2. Here we can proceed logically. What if the two numbers less than 20 are 19? Then their sum is 38 and the third number must be at least 33. Do you see why? If the two numbers are less than 19, then the third number will have to be more than 33. Therefore, this statement is true.

Investigation 3.1k – *Number Sense with Addition*

B. Take no more than 5 to 10 seconds to determine whether this answer is reasonable. Make your determination without doing any pencil-and-paper work.

$$\begin{array}{r} 6{,}563 \\ 4{,}448 \\ +\ 7{,}203 \\ \hline 17{,}241 \end{array}$$

Investigation 3.1k – *Discussion*

If we look at the leading digits, we see that they add up to 17 (meaning 17,000).

A very quick look at the rest of the amounts lets us quickly see that the answer must be greater than 17,214, and thus this answer is not reasonable.

Investigation 3.1k – *Number Sense with Addition*

C. Place > or < in the circle. Quickly look at the numbers on the left and right of the circle. Determine which symbol is appropriate.

$$563 + 924 + 723 \bigcirc 842 + 646 + 558$$

Investigation 3.1k – *Discussion*

The leading-digit method is useful again: $5 + 9 + 7 = 21$, whereas $8 + 6 + 5 = 19$; therefore,

$$563 + 924 + 723 > 842 + 646 + 558.$$