CHAPTER 9

Quadratic Equations and Inequalities

9.3 Equations That Are Reducible to Quadratic Equations

Objectives

1 Equations that are quadratic in form

2 Radical equations

3 Fractional equations

Equations that are quadratic in form

Equations that are quadratic in form

Certain equations that are not quadratic equations can be expressed in quadratic form by making suitable substitutions.

An equation is **quadratic in form** if it can be written as
$au^2 + bu + c = 0$.

To see that the equation at the right is quadratic in form, let $x^2 = u$. Replace x^2 by u. The equation is quadratic in form.

$$x^4 - 4x^2 - 5 = 0$$
$$(x^2)^2 - 4(x^2) - 5 = 0$$
$$u^2 - 4u - 5 = 0$$

Equations that are quadratic in form

The key to recognizing equations that are quadratic in form is that when the equation is written in standard form, the exponent on one variable term is $\frac{1}{2}$ the exponent on the other variable term.

Example 1

Solve: **A.** $x^4 + x^2 - 12 = 0$ **B.** $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 3 = 0$

Solution:

A. $x^4 + x^2 - 12 = 0$ — The equation is quadratic in form.

$(x^2)^2 + (x^2) - 12 = 0$

$u^2 + u - 12 = 0$ — Let $x^2 = u$.

$(u - 3)(u + 4) = 0$ — Solve for u by factoring.

Example 1 – *Solution*

cont'd

$$u - 3 = 0 \qquad u + 4 = 0$$

$$u = 3 \qquad u = -4$$

$$x^2 = 3 \qquad x^2 = -4$$

Replace *u* by x^2.

$$\sqrt{x^2} = \sqrt{3} \qquad \sqrt{x^2} = \sqrt{-4}$$

Solve for *x* by taking square roots.

$$x = \pm\sqrt{3} \qquad x = \pm 2i$$

The solutions are $\sqrt{3}$, $-\sqrt{3}$, 2*i*, and –2*i*.

Example 1 – *Solution*

cont'd

B. $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 3 = 0$ — The equation is quadratic in form.

$(x^{\frac{1}{3}})^2 - 2(x^{\frac{1}{3}}) - 3 = 0$

$u^2 - 2u - 3 = 0$ — Let $x^{\frac{1}{3}} = u$.

$(u - 3)(u + 1) = 0$ — Solve for u by factoring.

Example 1 – *Solution*

cont'd

$$u - 3 = 0 \qquad u + 1 = 0$$

$$u = 3 \qquad u = -1$$

$$x^{\frac{1}{3}} = 3 \qquad x^{\frac{1}{3}} = -1$$

Replace *u* by $x^{\frac{1}{3}}$.

$$\left(x^{\frac{1}{3}}\right)^3 = 3^3 \qquad \left(x^{\frac{1}{3}}\right)^3 = (-1)^3$$

Solve for *x* by cubing both sides of the equation.

$$x = 27 \qquad x = -1$$

The solutions are 27 and –1.

Radical equations

Radical equations

Certain equations containing a radical can be solved by first solving the equation for the radical expression and then squaring each side of the equation.

Remember that when each side of an equation has been squared, the resulting equation may have an extraneous solution. Therefore, the solutions of a radical equation must be checked.

Example 2

Solve: $\sqrt{3x + 7} - x = 3$

Solution:

$$\sqrt{3x + 7} - x = 3$$

$$\sqrt{3x + 7} = x + 3$$ Solve for the radical expression.

$$(\sqrt{3x + 7})^2 = (x + 3)^2$$ Square each side of the equation.

$$3x + 7 = x^2 + 6x + 9$$ Simplify.

$$0 = x^2 + 3x + 2$$ Write the equation in standard form.

$$0 = (x + 2)(x + 1)$$ Factor.

Example 2 – *Solution*

cont'd

$$x + 2 = 0 \qquad x + 1 = 0$$

Use the Principle of Zero Products.

$$x = -2 \qquad x = -1$$

Check:

$$\begin{array}{r|l} \multicolumn{2}{r}{\sqrt{3x + 7} - x = 3} \\ \hline \sqrt{3(-2) + 7} - (-2) & 3 \\ \sqrt{1} + 2 & \\ 1 + 2 & \\ \multicolumn{2}{r}{3 = 3} \end{array} \qquad \begin{array}{r|l} \multicolumn{2}{r}{\sqrt{3x + 7} - x = 3} \\ \hline \sqrt{3(-1) + 7} - (-1) & 3 \\ \sqrt{4} + 1 & \\ 2 + 1 & \\ \multicolumn{2}{r}{3 = 3} \end{array}$$

–2 and –1 check as solutions. The solutions are –2 and –1.

Radical equations

If an equation contains more than one radical, the procedure of solving for the radical expression and squaring each side of the equation may have to be repeated.

Example 3

Solve: $\sqrt{2x + 5} - \sqrt{x + 2} = 1$

Solution:

$$\sqrt{2x + 5} - \sqrt{x + 2} = 1$$

$$\sqrt{2x + 5} = \sqrt{x + 2} + 1$$

Solve for one of the radical expressions.

$$(\sqrt{2x + 5})^2 = (\sqrt{x + 2} + 1)^2$$

Square each side of the equation.

$$2x + 5 = x + 2 + 2\sqrt{x + 2} + 1$$

Simplify.

$$2x + 5 = x + 2\sqrt{x + 2} + 3$$

Example 3 – *Solution*

cont'd

$$x + 2 = 2\sqrt{x + 2}$$

Solve for the radical expression.

$$(x + 2)^2 = (2\sqrt{x + 2})^2$$

Square each side of the equation.

$$x^2 + 4x + 4 = 4(x + 2)$$

Simplify.

$$x^2 + 4x + 4 = 4x + 8$$

$$x^2 - 4 = 0$$

Write the equation in standard form.

Example 3 – *Solution*

cont'd

$$(x + 2)(x - 2) = 0$$

Factor.

$$x + 2 = 0 \qquad x - 2 = 0$$

Use the Principle of Zero Products.

$$x = -2 \qquad x = 2$$

The solutions are –2 and 2.

As shown at the left, -2 and 2 check as solutions.

Fractional equations

Fractional equations

After each side of a fractional equation has been multiplied by the LCD, the resulting equation is sometimes a quadratic equation.

The solutions to the resulting equation must be checked, because multiplying each side of an equation by a variable expression may produce an equation that has a solution that is not a solution of the original equation.

Example 4

Solve: $\dfrac{18}{2a - 1} + 3a = 17$

Solution:

$$\frac{18}{2a - 1} + 3a = 17$$

The LCD is $2a - 1$.

$$(2a - 1)\left(\frac{18}{2a - 1} + 3a\right) = (2a - 1)17$$

$$(2a - 1)\frac{18}{2a - 1} + (2a - 1)(3a) = (2a - 1)17$$

$$18 + 6a^2 - 3a = 34a - 17$$

Example 4 – *Solution*

cont'd

$$6a^2 - 37a + 35 = 0$$

Write the equation in standard form.

$$(6a - 7)(a - 5) = 0$$

Solve for *a* by factoring.

$$6a - 7 = 0 \qquad a - 5 = 0$$

$$6a = 7 \qquad a = 5$$

$$a = \frac{7}{6}$$

$\frac{7}{6}$ and 5 check as solutions.

The solutions are $\frac{7}{6}$ and 5.