Quadratic Equations and Inequalities

CHAPTER 9

9.2 Solving Quadratic Equations by Completing the Square and by Using the Quadratic Formula

Objectives

1 Solve quadratic equations by completing the square

2 Solve quadratic equations by using the quadratic formula

Solve quadratic equations by completing the square

Solve quadratic equations by completing the square

A perfect-square trinomial is the square of a binomial. Some examples of perfect-square trinomials follow.

Perfect-square trinomial		Square of a binomial
$x^2 + 8x + 16$	$=$	$(x + 4)^2$
$x^2 - 10x + 25$	$=$	$(x - 5)^2$
$x^2 - 5x + \frac{25}{4}$	$=$	$\left(x - \frac{5}{2}\right)^2$

Solve quadratic equations by completing the square

For each perfect-square trinomial, the square of $\frac{1}{2}$ the coefficient of *x* equals the constant term.

$$\left(\frac{1}{2}\,\textbf{coefficient of } x\right)^2 = \textbf{Constant term}$$

$$x^2 + 8x + 16, \qquad \left(\frac{1}{2}\cdot 8\right)^2 = 16$$

$$x^2 - 10x + 25, \qquad \left[\frac{1}{2}(-10)\right]^2 = 25$$

$$x^2 - 5x + \frac{25}{4}, \qquad \left(\frac{1}{2}(-5)\right)^2 = \frac{25}{4}$$

To **complete the square** on $x^2 + bx$, add $\left(\frac{1}{2}b\right)^2$ to $x^2 + bx$.

Solve quadratic equations by completing the square

Not all quadratic equations can be solved by factoring, but any quadratic equation can be solved by completing the square. Add to each side of the equation the term that completes the square.

Rewrite the equation in the form $(x + a)^2 = b$.

Then take the square root of each side of the equation.

When a, the coefficient of the x^2 term, is not 1, divide each side of the equation by a before completing the square.

Example 1

Solve by completing the square.

A. $4x^2 - 8x + 1 = 0$ **B.** $x^2 + 4x + 5 = 0$

Solution:

A. $4x^2 - 8x + 1 = 0$

$$4x^2 - 8x = -1$$

Subtract 1 from each side of the equation.

$$\frac{4x^2 - 8x}{4} = \frac{-1}{4}$$

The coefficient of the x^2 term must be 1. Divide each side of the equation by 4.

$$x^2 - 2x = -\frac{1}{4}$$

Example 1 – *Solution*

cont'd

$$x^2 - 2x + 1 = -\frac{1}{4} + 1$$

Complete the square. $\left[\frac{1}{2}(-2)\right]^2 = 1$

$$(x - 1)^2 = \frac{3}{4}$$

Factor the perfect-square trinomial.

$$\sqrt{(x - 1)^2} = \sqrt{\frac{3}{4}}$$

Take the square root of each side of the equation.

$$|x - 1| = \frac{\sqrt{3}}{2}$$

Simplify.

$$x - 1 = \pm\frac{\sqrt{3}}{2}$$

Example 1 – *Solution*

cont'd

$$x - 1 = \frac{\sqrt{3}}{2} \qquad x - 1 = -\frac{\sqrt{3}}{2}$$

Solve for *x*.

$$x = 1 + \frac{\sqrt{3}}{2} \qquad x = 1 - \frac{\sqrt{3}}{2}$$

$$x = \frac{2 + \sqrt{3}}{2} \qquad x = \frac{2 - \sqrt{3}}{2}$$

The solutions are $\frac{2 + \sqrt{3}}{2}$ and $\frac{2 - \sqrt{3}}{2}$.

Example 1 – *Solution*

cont'd

B. $x^2 + 4x + 5 = 0$

$$x^2 + 4x = -5$$

Subtract 5 from each side of the equation.

$$x^2 + 4x + 4 = -5 + 4$$

Complete the square.

$$(x + 2)^2 = -1$$

Factor the perfect-square trinomial.

$$\sqrt{(x + 2)^2} = \sqrt{-1}$$

Take the square root of each side of the equation.

$$|x + 2| = i$$

Simplify.

$$x + 2 = \pm i$$

Example 1 – *Solution*

cont'd

$x + 2 = i \qquad x + 2 = -i$ **Solve for *x*.**

$x = -2 + i \qquad x = -2 - i$

The solutions are $-2 + i$ and $-2 - i$.

Solve quadratic equations by using the quadratic formula

Solve quadratic equations by using the quadratic formula

A general formula known as the **quadratic formula** can be derived by applying the method of completing the square to the standard form of a quadratic equation.

This formula can be used to solve any quadratic equation.

QUADRATIC FORMULA

The solutions of $ax^2 + bx + c = 0$, $a \neq 0$, are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula is frequently written in the form

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 2

Solve by using the quadratic formula.

A. $4x^2 + 12x + 9 = 0$ **B.** $2x^2 - x + 5 = 0$

Solution:

A. $4x^2 + 12x + 9 = 0$

$a = 4, b = 12, c = 9$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-12 \pm \sqrt{12^2 - 4 \cdot 4 \cdot 9}}{2 \cdot 4}$$

Replace a, b, and c in the quadratic formula by their values. Then simplify.

Example 2 – *Solution*

cont'd

$$= \frac{-12 \pm \sqrt{144 - 144}}{8}$$

$$= \frac{-12 \pm \sqrt{0}}{8}$$

The equation has a double root.

$$= \frac{-12}{8}$$

$$= -\frac{3}{2}$$

The solution is $-\frac{3}{2}$.

Example 2 – *Solution*

cont'd

B. $2x^2 - x + 5 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

***a* = 2, *b* = −1, *c* = 5**

$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 2 \cdot 5}}{2 \cdot 2}$$

Replace *a*, *b*, and *c* in the quadratic formula by their values. Then simplify.

$$= \frac{1 \pm \sqrt{1 - 40}}{4}$$

Example 2 – *Solution*

cont'd

$$= \frac{1 \pm \sqrt{-39}}{4}$$

$$= \frac{1 \pm i\sqrt{39}}{4}$$

The solutions are $\frac{1}{4} + \frac{\sqrt{39}}{4}i$ and $\frac{1}{4} - \frac{\sqrt{39}}{4}i$.

Solve quadratic equations by using the quadratic formula

In the quadratic formula, the quantity $\boldsymbol{b^2 - 4ac}$ is called the **discriminant.**

When a, b, and c are real numbers, the discriminant determines whether a quadratic equation will have a double root, two real number solutions that are not equal, or two complex number solutions.

EFFECT OF THE DISCRIMINANT ON THE SOLUTIONS OF A QUADRATIC EQUATION

1. If $b^2 - 4ac = 0$, the equation has one real number solution, a double root.
2. If $b^2 - 4ac > 0$, the equation has two real number solutions that are not equal.
3. If $b^2 - 4ac < 0$, the equation has two complex number solutions.

Example 3

Use the discriminant to determine whether $4x^2 - 2x + 5 = 0$ has one real number solution, two real number solutions, or two complex number solutions.

Solution:

$$\begin{aligned} b^2 - 4ac &= (-2)^2 - 4(4)(5) \qquad && a = 4, b = -2, c = 5 \\ &= 4 - 80 \\ &= -76 \end{aligned}$$

$$-76 < 0 \qquad \text{The discriminant is less than 0.}$$

The equation has two complex number solutions.