CHAPTER 8

Rational Exponents and Radicals

8.5 Complex Numbers

Objectives

1 Simplify complex numbers

2 Add and subtract complex numbers

3 Multiply complex numbers

4 Divide complex numbers

Simplify complex numbers

Simplify complex numbers

The radical expression $\sqrt{-4}$ is not a real number because there is no real number whose square is –4. However, the solution of an algebraic equation is sometimes the square root of a negative number.

During the late 17th century, a new number, called an *imaginary number*, was defined so that a negative number would have a square root. The letter *i* was chosen to represent the number whose square is –1.

$$i^2 = -1$$

An imaginary number is defined in terms of *i.*

Simplify complex numbers

DEFINITION OF AN IMAGINARY NUMBER

If a is a positive real number, then the principal square root of $-a$ is the **imaginary number** $i\sqrt{a}$. This can be written

$$\sqrt{-a} = i\sqrt{a}$$

When $a = 1$, we have $\sqrt{-1} = i$.

EXAMPLES

1. $\sqrt{-16} = i\sqrt{16} = 4i$

2. $\sqrt{-21} = i\sqrt{21}$

It is customary to write *i* in front of a radical to avoid confusing $\sqrt{a}\,i$ with $\sqrt{ai}$.

Example 1

Simplify: $3\sqrt{-20}$

Solution:

$$3\sqrt{-20} = 3i\sqrt{20}$$

$$= 3i(2\sqrt{5})$$

$$= 6i\sqrt{5}$$

Simplify complex numbers

The set containing the real numbers and the imaginary numbers is called the set of complex numbers.

DEFINITION OF A COMPLEX NUMBER

A **complex number** is a number of the form $\boldsymbol{a + bi}$**,** where a and b are real numbers and $i = \sqrt{-1}$. The number a is the **real part** of the complex number, and b is the **imaginary part** of the complex number. A complex number written as $a + bi$ is in **standard form.**

EXAMPLES

1.	$3 + 4i$	Real part is 3; imaginary part is 4.
2.	$5 - 2i\sqrt{7}$	Real part is 5; imaginary part is $-2\sqrt{7}$.
3.	5	Real part is 5; imaginary part is 0, because $5 = 5 + 0i$.
4.	$-4i$	Real part is 0; imaginary part is -4, because $-4i = 0 - 4i$.
5.	$\dfrac{2 + 3i}{5} = \dfrac{2}{5} + \dfrac{3}{5}i$	Real part is $\frac{2}{5}$; imaginary part is $\frac{3}{5}$.

Simplify complex numbers

Complex numbers $a + bi$	Real numbers $a + 0i$	A **real number** is a complex number in which $b = 0$.
	Imaginary numbers $0 + bi$	An **imaginary number** is a complex number in which $a = 0$.

Example 2

Write $\dfrac{4 + \sqrt{-20}}{6}$ in standard form.

Solution:

$$\frac{4 + \sqrt{-20}}{6} = \frac{4 + i\sqrt{20}}{6}$$ Write $\sqrt{-20}$ as $i\sqrt{20}$.

$$= \frac{4 + 2i\sqrt{5}}{6}$$ Simplify the radical.

$$= \frac{\overset{1}{\cancel{2}}(2 + i\sqrt{5})}{\underset{1}{\cancel{2}} \cdot 3}$$

Example 2 – *Solution*

cont'd

$$= \frac{2 + i\sqrt{5}}{3}$$

Factor and simplify.

$$= \frac{2}{3} + \frac{\sqrt{5}}{3}i$$

Write in standard form.

Add and subtract complex numbers

Add and subtract complex numbers

ADDITION AND SUBTRACTION OF COMPLEX NUMBERS

To add two complex numbers, add the real parts and add the imaginary parts. To subtract two complex numbers, subtract the real parts and subtract the imaginary parts.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Example 3

Add or subtract.

A. $(3 + 2i) + (6 - 5i)$ **B.** $(-2 + 6i) - (4 - 3i)$

Solution:

A. $(3 + 2i) + (6 - 5i)$

$= (3 + 6) + (2 - 5)i$ **Add the real parts and add the imaginary parts.**

$= 9 - 3i$

Example 3 – *Solution*

cont'd

B. $(-2 + 6i) - (4 - 3i)$

$= (-2 - 4) + [6 - (-3)]i$ Subtract the real parts and subtract the imaginary parts.

$= -6 + 9i$

Multiply complex numbers

Multiply complex numbers

When we multiply complex numbers, the term i^2 is frequently a part of the product. Recall that $i^2 = -1$.

Example 4

Multiply.

A. $(3 - 4i)(2 + 5i)$ **B.** $\left(\frac{9}{10} + \frac{3}{10}i\right)\left(1 - \frac{1}{3}i\right)$

C. $(4 + 5i)(4 - 5i)$ **D.** $(6 + i)^2$

Solution:

A. $(3 - 4i)(2 + 5i)$

$= 6 + 15i - 8i - 20i^2$ **Use the FOIL method.**

$= 6 + 7i - 20i^2$ **Combine like terms.**

$= 6 + 7i - 20(-1)$ **Replace i^2 by -1.**

Example 4 – *Solution*

cont'd

$$= 26 + 7i$$

Write the answer in the form $a + bi$.

B. $\left(\frac{9}{10} + \frac{3}{10}i\right)\left(1 - \frac{1}{3}i\right)$

$$= \frac{9}{10} - \frac{3}{10}i + \frac{3}{10}i - \frac{1}{10}i^2$$

Use the FOIL method.

$$= \frac{9}{10} - \frac{1}{10}i^2$$

Combine like terms.

$$= \frac{9}{10} - \frac{1}{10}(-1)$$

Replace i^2 by -1.

Example 4 – *Solution*

cont'd

$$= \frac{9}{10} + \frac{1}{10}$$ **Simplify.**

$$= 1$$

c. $(4 + 5i)(4 - 5i)$

$$= 16 - 20i + 20i - 25i^2$$ **Use the FOIL method.**

$$= 16 - 25i^2$$

$$= 16 - 25(-1)$$

Example 4 – *Solution*

cont'd

$$= 16 + 25$$

$$= 41$$

D. $(6 + i)^2 = 36 + 12i + i^2$

$(6 + i)^2 = (6 + i)(6 + i)$

$$= 36 + 12i + (-1)$$

$$= 35 + 12i$$

Divide complex numbers

Divide complex numbers

A fraction containing one or more complex numbers is in simplest form when no imaginary number remains in the denominator.

Example 5

Divide. **A.** $\frac{5}{4i}$ **B.** $\frac{2+7i}{-14i}$

Solution:

A. $\frac{5}{4i} = \frac{5}{4i} \cdot \frac{i}{i}$

$= \frac{5i}{4i^2}$

Multiply the expression by 1 in the form $\frac{i}{i}$.

$= \frac{5i}{4(-1)}$

$= -\frac{5}{4}i$

Replace i^2 by -1. Then simplify.

Example 5 – *Solution*

cont'd

B. $$\frac{2+7i}{-14i} = \frac{2+7i}{-14i}\cdot\frac{i}{i}$$

Multiply the expression by 1 in the form $\frac{i}{i}$.

$$= \frac{2i+7i^2}{-14i^2}$$

$$= \frac{2i+7(-1)}{-14(-1)}$$

Replace i^2 by -1. Then simplify.

$$= \frac{-7+2i}{14}$$

$$= -\frac{1}{2}+\frac{1}{7}i$$

Write the answer in the form $a + bi$.

Divide complex numbers

CONJUGATE OF A COMPLEX NUMBER

The **conjugate** of $a + bi$ is $a - bi$, and the conjugate of $a - bi$ is $a + bi$. The product of the conjugates is $(a + bi)(a - bi) = a^2 + b^2$.

EXAMPLES

1. The conjugate of $2 + 5i$ is $2 - 5i$. The product of the conjugates is $(2 + 5i)(2 - 5i) = 2^2 + 5^2 = 29$.
2. The conjugate of $3 - 4i$ is $3 + 4i$. The product of the conjugates is $(3 - 4i)(3 + 4i) = 3^2 + 4^2 = 25$.
3. The conjugate of $-5 + i$ is $-5 - i$. The product of the conjugates is $(-5 + i)(-5 - i) = (-5)^2 + 1^2 = 26$.

The conjugate of a complex number is used to divide complex numbers when the denominator is of the form *a* + *bi*.

Example 6

Divide. **A.** $\dfrac{13i}{3 + 2i}$ **B.** $\dfrac{5 + 3i}{4 + 2i}$

Solution:

A. $\dfrac{13i}{3 + 2i} = \dfrac{13i}{3 + 2i} \cdot \dfrac{3 - 2i}{3 - 2i}$

Multiply the numerator and denominator by $3 - 2i$, the conjugate of the denominator.

$$= \frac{13i(3 - 2i)}{3^2 + 2^2}$$

$$= \frac{39i - 26i^2}{9 + 4}$$

Example 6 – *Solution*

cont'd

$$= \frac{39i - 26(-1)}{13}$$

Replace i^2 by -1. Then simplify.

$$= \frac{26 + 39i}{13}$$

$$= 2 + 3i$$

Write the answer in the form $a + bi$.

Example 6 – *Solution*

cont'd

B. $$\frac{5+3i}{4+2i} = \frac{5+3i}{4+2i}\cdot\frac{4-2i}{4-2i}$$

Multiply the numerator and denominator by $4 - 2i$, the conjugate of the denominator.

$$= \frac{20-10i+12i-6i^2}{4^2+2^2}$$

$$= \frac{20+2i-6(-1)}{16+4}$$

Replace i^2 by -1. Then simplify.

$$= \frac{20+2i+6}{20}$$

Example 6 – *Solution*

cont'd

$$= \frac{26 + 2i}{20}$$

$$= \frac{13}{10} + \frac{1}{10}i$$

Write the answer in the form *a* + *bi*.