CHAPTER

8

Rational Exponents and Radicals

8.2

Operations on Radical Expressions

Objectives

1 Simplify radical expressions

2 Add and subtract radical expressions

3 Multiply radical expressions

4 Divide radical expressions

Simplify radical expressions

Simplify radical expressions

A radical expression is not in simplest form if the radicand contains a factor that is a perfect power of the index. Here are some examples.

$\sqrt{32}$ is not in simplest form. $32 = 16 \cdot 2$, so $16 = 4^2$ is a perfect square factor of 32.

$\sqrt[3]{24}$ is not in simplest form. $24 = 8 \cdot 3$, so $8 = 2^3$ is a perfect cube factor of 24.

Simplify radical expressions

The Product Property of Radicals is used to write a radical expression in simplest form.

PRODUCT PROPERTY OF RADICALS

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$.

Example 1

Simplify: $\sqrt[4]{32x^6y^9z^2}$

Solution:

$$\sqrt[4]{32x^6y^9z^2} = \sqrt[4]{16x^4y^8 \cdot 2x^2yz^2}$$

Write the radicand as the product of a perfect fourth power and a factor that does not contain a perfect fourth power.

$$= \sqrt[4]{16x^4y^8} \cdot \sqrt[4]{2x^2yz^2}$$

Use the Product Property of Radicals.

$$= 2xy^2\sqrt[4]{2x^2yz^2}$$

Simplify.

Add and subtract radical expressions

Add and subtract radical expressions

The Distributive Property is used to simplify the sum or difference of radical expressions that have the same radicand and the same index. For example,

$$3\sqrt{5} + 8\sqrt{5} = (3 + 8)\sqrt{5} = 11\sqrt{5}$$

$$2\sqrt[3]{3x} - 9\sqrt[3]{3x} = (2 - 9)\sqrt[3]{3x} = -7\sqrt[3]{3x}$$

Add and subtract radical expressions

Radical expressions that are in simplest form and have unlike radicands or different indices cannot be simplified by the Distributive Property.

The following expressions cannot be simplified by the Distributive Property.

$3\sqrt[4]{2} - 6\sqrt[4]{3}$ **The radicands are different.**

$2\sqrt[4]{4x} + 3\sqrt[3]{4x}$ **The indices are different.**

Example 2

A. Add: $5\sqrt{20a^6b^3} + 4b\sqrt{125a^6b}$

B. Subtract: $2x\sqrt[3]{16y^7} - 4y\sqrt[3]{16x^3y^4}$

Solution:

A. $5\sqrt{20a^6b^3} + 4b\sqrt{125a^6b}$

$= 5\sqrt{4a^6b^2 \cdot 5b} + 4b\sqrt{25a^6 \cdot 5b}$ Use the Product Property of Radicals to simplify each radical expression.

$= 5\sqrt{4a^6b^2} \cdot \sqrt{5b} + 4b\sqrt{25a^6} \cdot \sqrt{5b}$

$= 5(2a^3b)\sqrt{5b} + 4b(5a^3)\sqrt{5b}$ $\sqrt{4a^6b^2} = 2a^3b$; $\sqrt{25a^6} = 5a^3$

$= 10a^3b\sqrt{5b} + 20a^3b\sqrt{5b}$ Simplify.

$= 30a^3b\sqrt{5b}$ Combine like terms.

Example 2 – *Solution*

cont'd

B. $2x\sqrt[3]{16y^7} - 4y\sqrt[3]{16x^3y^4}$

$= 2x\sqrt[3]{8y^6 \cdot 2y} - 4y\sqrt[3]{8x^3y^3 \cdot 2y}$ — Use the Product Property of Radicals to simplify each radical expression.

$= 2x\sqrt[3]{8y^6} \cdot \sqrt[3]{2y} - 4y\sqrt[3]{8x^3y^3} \cdot \sqrt[3]{2y}$

$= 2x(2y^2)\sqrt[3]{2y} - 4y(2xy)\sqrt[3]{2y}$ — $\sqrt[3]{8y^6} = 2y^2$; $\sqrt[3]{8x^3y^3} = 2xy$

$= 4xy^2\sqrt[3]{2y} - 8xy^2\sqrt[3]{2y}$ — Simplify.

$= -4xy^2\sqrt[3]{2y}$ — Combine like terms.

Multiply radical expressions

Multiply radical expressions

The Product Property of Radicals is used to multiply radical expressions with the same index.

Example 3

Multiply. **A.** $(3\sqrt{5})^2$ **B.** $\sqrt{2xy}\sqrt{6x}$ **C.** $\sqrt[3]{9a^2b}\sqrt[3]{18a^5b^2}$

Solution:

A. $(3\sqrt{5})^2 = (3\sqrt{5})(3\sqrt{5})$ — Use the Product Property of Radicals to multiply the radicands.

$= 9\sqrt{25}$

$= 9(5) = 45$ — Simplify.

B. $\sqrt{2xy}\sqrt{6x} = \sqrt{12x^2y}$ — Use the Product Property of Radicals.

$= \sqrt{4x^2}\sqrt{3y}$

$= 2x\sqrt{3y}$ — Simplify.

Example 3 – *Solution*

cont'd

c. $\sqrt[3]{9a^2b}\sqrt[3]{18a^5b^2} = \sqrt[3]{162a^7b^3}$ Use the Product Property of Radicals.

$= \sqrt[3]{27a^6b^3}\sqrt[3]{6a}$

$= 3a^2b\sqrt[3]{6a}$ Simplify.

Multiply radical expressions

When each of the radical expressions being multiplied contains two terms, use FOIL

Example 5

Multiply. **A.** $(2\sqrt{3} - 5)(7\sqrt{3} + 2)$ **B.** $(3\sqrt{x} - 2\sqrt{y})^2$

Solution:

A. $(2\sqrt{3} - 5)(7\sqrt{3} + 2)$

$= 14\sqrt{9} + 4\sqrt{3} - 35\sqrt{3} - 10$ **Use FOIL.**

$= 14(3) - 31\sqrt{3} - 10$ **Simplify $\sqrt{9}$ and combine like terms.**

$= 42 - 31\sqrt{3} - 10$

$= 32 - 31\sqrt{3}$

Example 5 – *Solution*

cont'd

B. $(3\sqrt{x} - 2\sqrt{y})^2$

$= (3\sqrt{x} - 2\sqrt{y})(3\sqrt{x} - 2\sqrt{y})$

$= 9\sqrt{x^2} - 6\sqrt{xy} - 6\sqrt{xy} + 4\sqrt{y^2}$ Use FOIL.

$= 9x - 12\sqrt{xy} + 4y$ Simplify $\sqrt{x^2}$ and $\sqrt{y^2}$. Combine like terms.

Divide radical expressions

Divide radical expressions

The Quotient Property of Radicals is used to divide radical expressions with the same index.

QUOTIENT PROPERTY OF RADICALS

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, and $b \neq 0$, then $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$.

SIMPLEST FORM OF A RADICAL EXPRESSION

A radical expression is in simplest form when *all* of the following conditions are met.

1. The radicand contains no factor greater than 1 that is a perfect power of the index.

2. There is no fraction under the radical sign.

3. No radical remains in the denominator of the radical expression.

Divide radical expressions

EXAMPLES

1. $\sqrt[3]{40}$ is not in simplest form. $8 = 2^3$ is a perfect cube factor of 40.

2. $\sqrt{\frac{2}{3}}$ is not in simplest form. There is a fraction under the radical sign.

3. $\frac{5}{\sqrt{6}}$ is not in simplest form. There is a radical in the denominator.

Condition 3 for the simplest form of a radical expression requires that no radical remain in the denominator of the radical expression. The procedure used to remove a radical from the denominator is called **rationalizing the denominator.**

Example 6

Simplify. **A.** $\sqrt{\dfrac{2}{3}}$ **B.** $\dfrac{5}{\sqrt{5x}}$ **C.** $\dfrac{3x}{\sqrt[3]{4x}}$

Solution:

A. $\sqrt{\dfrac{2}{3}} = \dfrac{\sqrt{2}}{\sqrt{3}}$

Use the Quotient Property of Radicals.

$= \dfrac{\sqrt{2}}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{6}}{\sqrt{9}}$

Rationalize the denominator by multiplying the numerator and denominator by $\sqrt{3}$.

$= \dfrac{\sqrt{6}}{3}$

Simplify.

Example 6 – *Solution*

cont'd

B. $$\frac{5}{\sqrt{5x}} = \frac{5}{\sqrt{5x}} \cdot \frac{\sqrt{5x}}{\sqrt{5x}}$$

Multiply the numerator and denominator by $\sqrt{5x}$.

$$= \frac{5\sqrt{5x}}{\sqrt{25x^2}}$$

$$= \frac{5\sqrt{5x}}{5x}$$

$$= \frac{\sqrt{5x}}{x}$$

Example 6 – *Solution*

cont'd

c. $$\frac{3x}{\sqrt[3]{4x}} = \frac{3x}{\sqrt[3]{4x}} \cdot \frac{\sqrt[3]{2x^2}}{\sqrt[3]{2x^2}}$$

Multiply the numerator and denominator by $\sqrt[3]{2x^2}$. Then $\sqrt[3]{4x} \cdot \sqrt[3]{2x^2} = \sqrt[3]{8x^3}$. $8x^3$ is a perfect cube.

$$= \frac{3x\sqrt[3]{2x^2}}{\sqrt[3]{8x^3}}$$

$$= \frac{3x\sqrt[3]{2x^2}}{2x}$$

$$= \frac{3\sqrt[3]{2x^2}}{2}$$

Divide radical expressions

To simplify a fraction that has a square-root radical expression with two terms in the denominator, multiply the numerator and denominator by the *conjugate* of the denominator.

DEFINITION OF CONJUGATE

The **conjugate** of $a + b$ is $a - b$, and the conjugate of $a - b$ is $a + b$. The product of conjugates is $(a + b)(a - b) = a^2 - b^2$.

EXAMPLES

1. The conjugate of $3 + \sqrt{7}$ is $3 - \sqrt{7}$. The product of the conjugates is

$$(3 + \sqrt{7})(3 - \sqrt{7}) = 3^2 - (\sqrt{7})^2 = 9 - 7 = 2$$

2. The conjugate of $\sqrt{5} - 6$ is $\sqrt{5} + 6$. The product of the conjugates is

$$(\sqrt{5} - 6)(\sqrt{5} + 6) = (\sqrt{5})^2 - 6^2 = 5 - 36 = -31$$

3. The conjugate of $-2 + 3\sqrt{2}$ is $-2 - 3\sqrt{2}$. The product of the conjugates is

$$(-2 + 3\sqrt{2})(-2 - 3\sqrt{2}) = (-2)^2 - (3\sqrt{2})^2 = 4 - (9 \cdot 2) = 4 - 18 = -14$$

4. The conjugate of $\sqrt{x} - \sqrt{y}$ is $\sqrt{x} + \sqrt{y}$. The product of the conjugates is

$$(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y}) = (\sqrt{x})^2 - (\sqrt{y})^2 = x - y$$

Example 7

Simplify. **A.** $\dfrac{4 - 3\sqrt{5}}{3 + 2\sqrt{5}}$ **B.** $\dfrac{\sqrt{3} + \sqrt{y}}{\sqrt{3} - \sqrt{y}}$

Solution:

A. $$\frac{4 - 3\sqrt{5}}{3 + 2\sqrt{5}} = \frac{4 - 3\sqrt{5}}{3 + 2\sqrt{5}} \cdot \frac{3 - 2\sqrt{5}}{3 - 2\sqrt{5}}$$

Multiply the numerator and denominator by the conjugate of the denominator.

$$= \frac{12 - 8\sqrt{5} - 9\sqrt{5} + 6\sqrt{25}}{3^2 - (2\sqrt{5})^2}$$

Use FOIL.
Use $(a + b)(a - b)$ $= a^2 - b^2$.

$$= \frac{12 - 17\sqrt{5} + 30}{9 - 20}$$

Simplify.
$6\sqrt{25} = 6(5) = 30$;
$(2\sqrt{5})^2 = 4(5) = 20$

$$= \frac{42 - 17\sqrt{5}}{-11} = -\frac{42 - 17\sqrt{5}}{11}$$

Example 7 – *Solution*

cont'd

B. $$\frac{\sqrt{3}+\sqrt{y}}{\sqrt{3}-\sqrt{y}} = \frac{\sqrt{3}+\sqrt{y}}{\sqrt{3}-\sqrt{y}} \cdot \frac{\sqrt{3}+\sqrt{y}}{\sqrt{3}+\sqrt{y}}$$

Multiply the numerator and denominator by the conjugate of the denominator.

$$= \frac{\sqrt{9}+\sqrt{3y}+\sqrt{3y}+\sqrt{y^2}}{(\sqrt{3})^2-(\sqrt{y})^2}$$

Use FOIL.

Use $(a-b)(a+b) = a^2 - b^2$.

$$= \frac{3+2\sqrt{3y}+y}{3-y}$$

Simplify.