Rational Exponents and Radicals

CHAPTER

8

8.1 Rational Exponents and Radical Expressions

Objectives

1 Simplify expressions with rational exponents

2 Write exponential expressions as radical expressions and radical expressions as exponential expressions

3 Simplify radical expressions that are roots of perfect powers

Simplify expressions with rational exponents

Simplify expressions with rational exponents

In this section, the definition of an exponent is extended beyond integers so that any rational number can be used as an exponent. The definition is expressed in such a way that the Rules of Exponents hold true for rational exponents.

DEFINITION OF $a^{\frac{1}{n}}$

If n is a positive integer, then $a^{\frac{1}{n}}$ is the number whose nth power is a.

EXAMPLES

1. $9^{\frac{1}{2}} = 3$ because $3^2 = 9$.
2. $64^{\frac{1}{3}} = 4$ because $4^3 = 64$.
3. $(-32)^{\frac{1}{5}} = -2$ because $(-2)^5 = -32$.

Simplify expressions with rational exponents

Using the definition of $a^{\frac{1}{n}}$ and the Rules of Exponents, it is possible to define any exponential expression that contains a rational exponent.

DEFINITION OF $a^{\frac{m}{n}}$

If m and n are positive integers and $a^{\frac{1}{n}}$ is a real number, then

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$$

Example 1

Simplify: **A.** $27^{\frac{2}{3}}$ **B.** $32^{-\frac{2}{5}}$

Solution:

A. $27^{\frac{2}{3}} = (3^3)^{\frac{2}{3}}$ Rewrite 27 as 3^3.

$= 3^{3\left(\frac{2}{3}\right)}$ Use the Rule for Simplifying a Power of an Exponential Expression.

$= 3^2$

$= 9$ Simplify.

Example 1 – *Solution*

cont'd

B. $32^{-\frac{2}{5}} = (2^5)^{-\frac{2}{5}}$ — Rewrite 32 as 2^5.

$= 2^{-2}$ — Use the Rule for Simplifying a Power of an Exponential Expression.

$= \dfrac{1}{2^2}$ — Use the Definition of a Negative Exponent.

$= \dfrac{1}{4}$ — Simplify.

Write exponential expressions as radical expressions and radical expressions as exponential expressions

Write exponential expressions as radical expressions and radical expressions as exponential expressions

$a^{\frac{1}{n}}$ is the number whose *n*th power is *a*. We can also say that $a^{\frac{1}{n}}$ is the ***n*th root of *a***.

*n*TH ROOT OF *a*

If a is a real number and n is a positive integer, then $\sqrt[n]{a} = a^{\frac{1}{n}}$.

In the expression $\sqrt[n]{a}$, the symbol $\sqrt{}$ is called the **radical,** n is the **index** of the radical, and a is the **radicand.**

EXAMPLES

1. $\sqrt[3]{7} = 7^{\frac{1}{3}}$

2. $\sqrt[5]{x} = x^{\frac{1}{5}}$

Write exponential expressions as radical expressions and radical expressions as exponential expressions

When n = 2, the radical expression represents a square root and the index 2 is usually not written.

An exponential expression with a rational exponent can be written as a radical expression.

WRITE $a^{\frac{m}{n}}$ AS A RADICAL EXPRESSION

If $a^{\frac{1}{n}}$ is a real number, then $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

EXAMPLES

1. $15^{\frac{2}{3}} = \sqrt[3]{15^2}$

2. $x^{\frac{4}{5}} = \sqrt[5]{x^4}$

3. $\sqrt[5]{2^3} = 2^{\frac{3}{5}}$

4. $\sqrt[6]{z^5} = z^{\frac{5}{6}}$

Example 3

Rewrite the exponential expression as a radical expression.

A. $(3x)^{\frac{2}{3}}$ **B.** $-2x^{\frac{2}{3}}$

Solution:

A. $(3x)^{\frac{2}{3}} = \sqrt[3]{(3x)^2}$

$= \sqrt[3]{9x^2}$

The denominator of the rational exponent is the index of the radical. The numerator is the power of the radicand.

B. $-2x^{\frac{2}{3}} = -2(x^2)^{\frac{1}{3}}$

$= -2\sqrt[3]{x^2}$

The -2 is not raised to the power.

Simplify radical expressions that are roots of perfect powers

Simplify radical expressions that are roots of perfect powers

Every positive number has two square roots, one a positive and one a negative number. For example, because $(5)^2 = 25$ and $(-5)^2 = 25$, there are two square roots of 25: 5 and –5.

The symbol $\sqrt{}$ is used to indicate the positive or **principal square root**. To indicate the negative square root of a number, a negative sign is placed in front of the radical.

The square root of zero is zero.

Simplify radical expressions that are roots of perfect powers

The square root of a negative number is not a real number, because the square of a real number must be positive.

The square root of a squared negative number is a positive number.

For any real number *a*, $\sqrt{a^2} = |a|$ and $-\sqrt{a^2} = -|a|$.
If *a* is a positive real number, then $\sqrt{a^2} = a$ and $(\sqrt{a})^2 = a$.

Simplify radical expressions that are roots of perfect powers

The cube root of a positive number is positive.

The cube root of a negative number is negative.

For any real number *a*, $\sqrt[3]{a^3}$ = *a.*

The following holds true for finding the *n*th root of a real number.

If *n* is an even integer, then $\sqrt[n]{a^n}$ = |*a*| and $-\sqrt[n]{a^n}$ = –|*a*|.
If *n* is an odd integer, then $\sqrt[n]{a^n}$ = *a*.

Simplify radical expressions that are roots of perfect powers

Note that a variable expression is a perfect power if the exponents on the factors are evenly divisible by the index of the radical.

The chart below shows roots of perfect powers. Knowledge of these roots is very helpful in simplifying radical expressions.

Square Roots		*Cube Roots*	*Fourth Roots*	*Fifth Roots*
$\sqrt{1} = 1$	$\sqrt{36} = 6$	$\sqrt[3]{1} = 1$	$\sqrt[4]{1} = 1$	$\sqrt[5]{1} = 1$
$\sqrt{4} = 2$	$\sqrt{49} = 7$	$\sqrt[3]{8} = 2$	$\sqrt[4]{16} = 2$	$\sqrt[5]{32} = 2$
$\sqrt{9} = 3$	$\sqrt{64} = 8$	$\sqrt[3]{27} = 3$	$\sqrt[4]{81} = 3$	$\sqrt[5]{243} = 3$
$\sqrt{16} = 4$	$\sqrt{81} = 9$	$\sqrt[3]{64} = 4$	$\sqrt[4]{256} = 4$	
$\sqrt{25} = 5$	$\sqrt{100} = 10$	$\sqrt[3]{125} = 5$	$\sqrt[4]{625} = 5$	

Simplify radical expressions that are roots of perfect powers

If a number is not a perfect power, its root can only be approximated; examples include $\sqrt{5}$ and $\sqrt[3]{3}$. These numbers are **irrational numbers**.

Their decimal representations never terminate or repeat.

$$\sqrt{5} = 2.2360679\ldots \qquad \sqrt[3]{3} = 1.4422495\ldots$$

Example 5

Simplify.

A. $\sqrt{49x^2}$ **B.** $\sqrt[3]{-125a^6b^9}$ **C.** $-\sqrt[4]{16a^4b^8}$

Solution:

A. $\sqrt{49x^2} = 7x$

The radicand is a perfect square. Divide the exponent by 2.

B. $\sqrt[3]{-125a^6b^9} = -5a^2b^3$

The radicand is a perfect cube. Divide each exponent by 3.

C. $-\sqrt[4]{16a^4b^8} = -2ab^2$

The radicand is a perfect fourth power. Divide each exponent by 4.