CHAPTER 7

Rational Expressions

7.2 Operations on Rational Expressions

Objectives

1 Multiply and divide rational expressions

2 Add and subtract rational expressions

Multiply and divide rational expressions

Multiply and divide rational expressions

MULTIPLY RATIONAL EXPRESSIONS

The product of two rational expressions is the product of the numerators of the expressions over the product of the denominators of the expressions.

EXAMPLES

1. $\dfrac{2x^2}{3y} \cdot \dfrac{6y^2}{x^3} = \dfrac{12x^2y^2}{3x^3y}$

- **Multiply the numerators. Multiply the denominators.**

$= \dfrac{4y}{x}$

- **Write the answer in simplest form.**

2. $\dfrac{x+2}{2x-6} \cdot \dfrac{5x-15}{3x+6} = \dfrac{x+2}{2(x-3)} \cdot \dfrac{5(x-3)}{3(x+2)}$

- **Factor the numerators and denominators.**

$= \dfrac{5\overset{1}{\cancel{(x+2)}}\overset{1}{\cancel{(x-3)}}}{6\underset{1}{\cancel{(x-3)}}\underset{1}{\cancel{(x+2)}}}$

- **Multiply the numerators. Multiply the denominators.**

$= \dfrac{5}{6}$

- **Write the answer in simplest form.**

Example 1

Multiply: $\dfrac{2x^2 - 5x - 3}{x^2 - 7x + 12} \cdot \dfrac{20 - x - x^2}{2x^2 + 13x + 6}$

Solution:

$$\frac{2x^2 - 5x - 3}{x^2 - 7x + 12} \cdot \frac{20 - x - x^2}{2x^2 + 13x + 6}$$

$$= \frac{(2x + 1)(x - 3)}{(x - 3)(x - 4)} \cdot \frac{(5 + x)(4 - x)}{(2x + 1)(x + 6)}$$

Factor the numerators and denominators.

$$= \frac{\overset{1}{\cancel{(2x + 1)}}\overset{1}{\cancel{(x - 3)}}(5 + x)\overset{-1}{\cancel{(4 - x)}}}{\underset{1}{\cancel{(x - 3)}}\underset{1}{\cancel{(x - 4)}}\underset{1}{\cancel{(2x + 1)}}(x + 6)}$$

Multiply the rational expressions. Recall that $\dfrac{4 - x}{x - 4} = \dfrac{-\overset{1}{\cancel{(x - 4)}}}{\underset{1}{\cancel{x - 4}}} = -1.$

$$= -\frac{x + 5}{x + 6}$$

Write the answer in simplest form.

Multiply and divide rational expressions

The **reciprocal of a rational expression** is the rational expression with the numerator and denominator interchanged.

$$\text{Rational expression} \quad \left\{ \begin{array}{cc} \dfrac{a}{b} & \dfrac{b}{a} \\[2ex] \dfrac{a^2 - 2y}{4} & \dfrac{4}{a^2 - 2y} \end{array} \right\} \quad \text{Reciprocal}$$

Multiply and divide rational expressions

DIVISION OF RATIONAL EXPRESSIONS

To divide rational expressions, multiply the dividend by the reciprocal of the divisor.

EXAMPLES

1. $\frac{5a^2b}{7x^2y} \div \frac{10a^3b^2}{9xy^2} = \frac{5a^2b}{7x^2y} \cdot \frac{9xy^2}{10a^3b^2}$

- Multiply by the reciprocal of the divisor.

$= \frac{45a^2bxy^2}{70a^3b^2x^2y}$

- Multiply.

$= \frac{9y}{14abx}$

- Write the answer in simplest form.

2. $\frac{3x+15}{5x^2} \div \frac{6x+30}{4x} = \frac{3x+15}{5x^2} \cdot \frac{4x}{6x+30}$

- Multiply by the reciprocal of the divisor.

$= \frac{3(x+5)}{5x^2} \cdot \frac{4x}{6(x+5)}$

- Factor the numerators and denominators.

$= \frac{12x\overset{1}{\cancel{(x+5)}}}{30x^2\underset{1}{\cancel{(x+5)}}} = \frac{2}{5x}$

- Multiply. Then write the answer in simplest form.

Example 2

Divide: **A.** $\dfrac{12x^2y^2 - 24xy^2}{5z^2} \div \dfrac{4x^3y - 8x^2y}{3z^4}$ **B.** $\dfrac{3y^2 - 10y + 8}{3y^2 + 8y - 16} \div \dfrac{2y^2 - 7y + 6}{2y^2 + 5y - 12}$

Solution:

A. $\dfrac{12x^2y^2 - 24xy^2}{5z^2} \div \dfrac{4x^3y - 8x^2y}{3z^4}$

$= \dfrac{12x^2y^2 - 24xy^2}{5z^2} \cdot \dfrac{3z^4}{4x^3y - 8x^2y}$

Multiply by the reciprocal of the divisor.

$= \dfrac{12xy^2(x - 2)}{5z^2} \cdot \dfrac{3z^4}{4x^2y(x - 2)}$

Factor the numerators and denominators.

$= \dfrac{36xy^2z^4\overset{1}{\cancel{(x - 2)}}}{20x^2yz^2\underset{1}{\cancel{(x - 2)}}} = \dfrac{9yz^2}{5x}$

Multiply. Then write the answer in simplest form.

B. $$\frac{3y^2 - 10y + 8}{3y^2 + 8y - 16} \div \frac{2y^2 - 7y + 6}{2y^2 + 5y - 12}$$

$$= \frac{3y^2 - 10y + 8}{3y^2 + 8y - 16} \cdot \frac{2y^2 + 5y - 12}{2y^2 - 7y + 6}$$

Multiply by the reciprocal of the divisor.

$$= \frac{(y - 2)(3y - 4)}{(3y - 4)(y + 4)} \cdot \frac{(y + 4)(2y - 3)}{(y - 2)(2y - 3)}$$

Factor the numerators and denominators.

$$= \frac{\overset{1}{\cancel{(y - 2)}}\overset{1}{\cancel{(3y - 4)}}\overset{1}{\cancel{(y + 4)}}\overset{1}{\cancel{(2y - 3)}}}{\underset{1}{\cancel{(3y - 4)}}\underset{1}{\cancel{(y + 4)}}\underset{1}{\cancel{(y - 2)}}\underset{1}{\cancel{(2y - 3)}}} = 1$$

Multiply. Then write the answer in simplest form.

Add and subtract rational expressions

Add and subtract rational expressions

ADD OR SUBTRACT RATIONAL EXPRESSIONS

To add two rational expressions *with the same denominator*, add the numerators and place the sum over the common denominator. To subtract two rational expressions *with the same denominator*, subtract the numerators and place the difference over the common denominator.

EXAMPLES

1. $$\frac{2a+b}{a^2-b^2}+\frac{a-4b}{a^2-b^2}=\frac{(2a+b)+(a-4b)}{a^2-b^2}$$

$$=\frac{3a-3b}{a^2-b^2}=\frac{3\overset{1}{\cancel{(a-b)}}}{(a+b)\underset{1}{\cancel{(a-b)}}}$$

$$=\frac{3}{a+b}$$

- **The denominators are the same. Add the numerators.**
- **Simplify. Write the fraction in simplest form.**

Add and subtract rational expressions

2. $\dfrac{7x - 12}{2x^2 + 5x - 12} - \dfrac{3x - 6}{2x^2 + 5x - 12}$

$= \dfrac{(7x - 12) - (3x - 6)}{2x^2 + 5x - 12}$

$= \dfrac{7x - 12 - 3x + 6}{2x^2 + 5x - 12}$

$= \dfrac{4x - 6}{2x^2 + 5x - 12}$

$= \dfrac{2(2x - 3)}{(2x - 3)(x + 4)}$

$= \dfrac{2\cancel{(2x - 3)}^{1}}{\cancel{(2x - 3)}_{1}(x + 4)} = \dfrac{2}{x + 4}$

- **The denominators are the same. Subtract the numerators.**
- **Simplify. Write the fraction in simplest form.**

Add and subtract rational expressions

For each example above, the denominators of the two expressions were the same. If the denominators of two expressions are not the same, both rational expressions must be expressed in terms of a *common denominator*.

A good common denominator to use is the least common multiple (LCM) of the denominators, also called the least common denominator (LCD).

Add and subtract rational expressions

The **LCM of two or more polynomials** is the polynomial of least degree that contains the factors of each polynomial.

To find the LCM, first factor each polynomial completely.

The LCM is the product of each factor the greatest number of times it occurs in any one factorization.

Example 3

Add: $\dfrac{4b}{a^2} + \dfrac{2a}{b^2} + \dfrac{3}{ab}$

Solution:

The LCD is a^2b^2. **Find the LCD.**

$$\frac{4b}{a^2} + \frac{2a}{b^2} + \frac{3}{ab} = \frac{4b}{a^2} \cdot \frac{b^2}{b^2} + \frac{2a}{b^2} \cdot \frac{a^2}{a^2} + \frac{3}{ab} \cdot \frac{ab}{ab}$$

Write each fraction in terms of the LCD.

$$= \frac{4b^3}{a^2b^2} + \frac{2a^3}{a^2b^2} + \frac{3ab}{a^2b^2}$$

Simplify.

$$= \frac{4b^3 + 2a^3 + 3ab}{a^2b^2}$$

Add the numerators. Place the sum over the common denominator.

Example 4

Subtract: $\dfrac{x}{2x-4} - \dfrac{4-x}{x^2-2x}$

Solution:

$$2x - 4 = 2(x-2)$$
$$x^2 - 2x = x(x-2)$$

The LCD is $2x(x-2)$.

Find the LCD.

$$\frac{x}{2x-4} - \frac{4-x}{x^2-2x}$$
$$= \frac{x}{2(x-2)} \cdot \frac{x}{x} - \frac{4-x}{x(x-2)} \cdot \frac{2}{2}$$

Write each fraction in terms of the LCD.

$$= \frac{x^2}{2x(x-2)} - \frac{8-2x}{2x(x-2)}$$

Example 4 – *Solution*

cont'd

$$= \frac{x^2 - (8 - 2x)}{2x(x - 2)}$$

Subtract the fractions.

$$= \frac{x^2 + 2x - 8}{2x(x - 2)}$$

Simplify.

$$= \frac{(x + 4)(x - 2)}{2x(x - 2)}$$

$$= \frac{(x + 4)\cancel{(x - 2)}^{1}}{2x\cancel{(x - 2)}_{1}} = \frac{x + 4}{2x}$$

Divide by the common factors.

Example 5

Simplify: **A.** $\dfrac{4}{x-3} + 3 - \dfrac{2x}{x-1}$

B. $\dfrac{6x-23}{2x^2+x-6} + \dfrac{3x}{2x-3} - \dfrac{5}{x+2}$

Solution:

A. The LCD is $(x-3)(x-1)$.

Find the LCD.

$$\frac{4}{x-3} + 3 - \frac{2x}{x-1}$$

$$= \frac{4}{x-3} \cdot \frac{x-1}{x-1} + \frac{3}{1} \cdot \frac{(x-3)(x-1)}{(x-3)(x-1)} - \frac{2x}{x-1} \cdot \frac{x-3}{x-3}$$

Write each fraction in terms of the LCD.

Example 5 – *Solution*

cont'd

$$= \frac{4x - 4}{(x - 3)(x - 1)} + \frac{3x^2 - 12x + 9}{(x - 3)(x - 1)} - \frac{2x^2 - 6x}{(x - 3)(x - 1)}$$

Simplify.

$$= \frac{(4x - 4) + (3x^2 - 12x + 9) - (2x^2 - 6x)}{(x - 3)(x - 1)}$$

Write the sum and difference over the common denominator.

$$= \frac{x^2 - 2x + 5}{(x - 3)(x - 1)}$$

Write the answer in simplest form.

Example 5 – *Solution*

cont'd

B. $2x^2 + x - 6 = (2x - 3)(x + 2)$

The LCD is $(2x - 3)(x + 2)$. **Find the LCD.**

$$\frac{6x - 23}{2x^2 + x - 6} + \frac{3x}{2x - 3} - \frac{5}{x + 2}$$

$$= \frac{6x - 23}{(2x - 3)(x + 2)} + \frac{3x}{2x - 3} \cdot \frac{x + 2}{x + 2} - \frac{5}{x + 2} \cdot \frac{2x - 3}{2x - 3}$$

Write each fraction in terms of the LCD.

$$= \frac{6x - 23}{(2x - 3)(x + 2)} + \frac{3x^2 + 6x}{(2x - 3)(x + 2)} - \frac{10x - 15}{(2x - 3)(x + 2)}$$

Simplify.

Example 5 – *Solution*

cont'd

$$= \frac{(6x - 23) + (3x^2 + 6x) - (10x - 15)}{(2x - 3)(x + 2)}$$

Write the sum and difference over the common denominator.

$$= \frac{6x - 23 + 3x^2 + 6x - 10x + 15}{(2x - 3)(x + 2)}$$

Simplify the numerator.

$$= \frac{3x^2 + 2x - 8}{(2x - 3)(x + 2)}$$

$$= \frac{(3x - 4)\overset{1}{\cancel{(x + 2)}}}{(2x - 3)\underset{1}{\cancel{(x + 2)}}} = \frac{3x - 4}{2x - 3}$$

Write the answer in simplest form.