Exponential and Logarithmic Functions

CHAPTER 11

11.5 Applications of Exponential and Logarithmic Functions

Objective

1 Application problems

Application problems

Application problems

A biologist places one single-celled bacterium in a culture, and each hour that particular species of bacterium divides into two bacteria.

After 1 h, there will be two bacteria.

After 2 h, each of the two bacteria will divide and there will be four bacteria.

After 3 h, each of the four bacteria will divide and there will be eight bacteria.

Application problems

The table below shows the number of bacteria in the culture after various intervals of time t, in hours. Values in this table could also be found by using the exponential equation $N = 2^t$.

Time, t	Number of Bacteria, N
0	1
1	2
2	4
3	8
4	16

Application problems

The equation $N = 2^t$ is an example of an **exponential growth equation.**

In general, any equation that can be written in the form $\boldsymbol{A = A_0 b^{kt}}$, where A is the size at time t, A_0 is the initial size, $b > 1$, and k is a positive real number, is an exponential growth equation.

These equations are important not only in population growth studies but also in physics, chemistry, psychology, and economics.

Application problems

Interest is the amount of money one pays (or receives) when borrowing (or investing) money.

Compound interest is interest that is computed not only on the original principal but also on the interest already earned. The compound interest formula is an exponential growth equation.

The **compound interest formula** is $\boldsymbol{P = A(1 + i)^n}$, where A is the original value of an investment, i is the interest rate per compounding period, n is the total number of compounding periods, and P is the value of the investment after n periods.

Application problems

Exponential decay can also be modeled by an exponential equation. One of the most common illustrations of exponential decay is the decay of a radioactive substance.

A radioactive isotope of cobalt has a half-life of approximately 5 years.

This means that one-half of any given amount of the cobalt isotope will disintegrate in 5 years. Suppose you begin with 10 mg of a cobalt isotope.

Application problems

The table below indicates the amount of the initial 10 mg of cobalt isotope that remains after various intervals of time *t*, in years. Values in this table could also be found by using the exponential equation $A = 10\left(\frac{1}{2}\right)^{\frac{t}{5}}$.

Time, t	Amount, A
0	10
5	5
10	2.5
15	1.25
20	0.625

The equation $A = 10\left(\frac{1}{2}\right)^{\frac{t}{5}}$ is an example of an **exponential decay equation.**

Example 1

Molybdenum-99 is a radioactive isotope used in medicine. An original 20-microgram sample of molybdenum-99 decays to 18 micrograms in 10 h. Find the half-life of molybdenum-99. Round to the nearest hour.

Strategy:

A_0, the original amount, is 20 micrograms. A, the final amount, is 18 micrograms. The time is 10 h.

To find the half-life, solve the exponential decay equation $A = A_0\left(\frac{1}{2}\right)^{\frac{t}{k}}$ for the half-life, *k*.

Example 1 – *Solution*

$$A = A_0\left(\frac{1}{2}\right)^{\frac{t}{k}}$$

Use the exponential decay equation.

$$18 = 20\left(\frac{1}{2}\right)^{\frac{10}{k}}$$

$A_0 = 20$, $A = 18$, $t = 10$

$$\frac{18}{20} = \left(\frac{1}{2}\right)^{\frac{10}{k}}$$

Solve for k.

$$\log\frac{18}{20} = \log\left(\frac{1}{2}\right)^{\frac{10}{k}}$$

$$\log\frac{18}{20} = \frac{10}{k}\log\frac{1}{2}$$

Example 1 – *Solution*

cont'd

$$k = \frac{10 \log \frac{1}{2}}{\log \frac{18}{20}}$$

$$k \approx 65.8$$

The half-life of molybdenum-99 is about 66 h.

Application problems

The first applications of logarithms (and the main reason why they were developed) were to reduce computational drudgery.

Today, with the widespread use of calculators and computers, the computational uses of logarithms have diminished.

However, a number of other applications of logarithms have emerged.

Application problems

A chemist measures the acidity or alkalinity of a solution by the formula $\mathbf{pH = -\log(H^{+})}$**,** where H^{+} is the concentration of hydrogen ions in the solution.

A neutral solution such as distilled water has a pH of 7, acids have a pH less than 7, and alkaline solutions (also called basic solutions) have a pH greater than 7.

Logarithmic functions are used to scale very large or very small numbers into numbers that are easier to comprehend.

Application problems

For instance, the *Richter scale* magnitude of an earthquake uses a logarithmic function to convert the intensity of shock waves I into a number M, which for most earthquakes is in the range of 0 to 10.

The intensity I of an earthquake is often given in terms of the constant I_0, where I_0 is the intensity of the smallest earthquake, called a **zero-level earthquake,** that can be measured on a seismograph near the earthquake's epicenter.

Application problems

An earthquake with an intensity I_0 has a Richter scale magnitude of $M = \log \frac{I}{I_0}$, where I_0 is the measure of a zero-level earthquake.

If you know the Richter scale magnitude of an earthquake, you can determine the intensity of the earthquake.

The percent of light that will pass through a substance is given by the equation **$\log P = -kd$,** where P is the percent of light, as a decimal, passing through the substance, k is a constant that depends on the substance, and d is the thickness of the substance in centimeters.

Example 2

Find the hydrogen ion concentration H^+ of orange juice that has a pH of 3.6.

Strategy:

To find the hydrogen ion concentration, replace pH by 3.6 in the equation $pH = -\log(H^+)$ and solve for H^+.

Example 2 – *Solution*

$$\begin{aligned} \text{pH} &= -\log(\text{H}^+) \\ 3.6 &= -\log(\text{H}^+) \\ -3.6 &= \log(\text{H}^+) \\ 10^{-3.6} &= \text{H}^+ \\ 0.00025 &\approx \text{H}^+ \end{aligned}$$

The hydrogen ion concentration is approximately 0.00025.