CHAPTER 11

Exponential and Logarithmic Functions

11.2 Introduction to Logarithms

Objectives

1 Write equivalent exponential and logarithmic equations

2 The properties of logarithms

Write equivalent exponential and logarithmic equations

Write equivalent exponential and logarithmic equations

The exponential function given by $y = b^x$ is a 1–1 function, it has an inverse function. To find that function, we follow the same procedure that we used to find the inverses of other functions.

$y = b^x$

$x = b^y$ Interchange x and y.

The equation $x = b^y$ says that y equals the power of b that produces x. That is, y is the logarithm base b of x. The inverse of the exponential function $y = b^x$ is a logarithmic function.

Write equivalent exponential and logarithmic equations

Here is a general definition of logarithm.

DEFINITION OF LOGARITHM

If $x > 0$ and b is a positive constant not equal to 1, then $y = \log_b x$ is equivalent to $b^y = x$.

Read $\log_b x$ as "the logarithm base *b* of *x*" or "the log base *b* of *x*."

Write equivalent exponential and logarithmic equations

The following table shows equivalent statements written in both exponential and logarithmic form.

Exponential Form		Logarithmic Form
$2^4 = 16$	$\Longleftrightarrow$	$\log_2 16 = 4$
$\left(\frac{2}{3}\right)^2 = \frac{4}{9}$	$\Longleftrightarrow$	$\log_{\frac{2}{3}}\left(\frac{4}{9}\right) = 2$
$10^{-1} = 0.1$	$\Longleftrightarrow$	$\log_{10}(0.1) = -1$
$b^a = x$	$\Longleftrightarrow$	$\log_b x = a$

Example 1

Write $4^5 = 1024$ in logarithmic form.

Solution:

$4^5 = 1024$ is equivalent to $\log_4 1024 = 5$.

Example 2

Write $\log_7 343 = 3$ in exponential form.

Solution:

$\log_7 343 = 3$ is equivalent to $7^3 = 343$.

Write equivalent exponential and logarithmic equations

In Example 2, 343 can be referred to as the *antilogarithm* base 7 of 3. If $\log_b M = N$, then N is the logarithm base b of M; M is the **antilogarithm** base b of N.

Recalling the equations $y = \log_b x$ and $x = b^y$ from the definition of a logarithm, note that because $b^y > 0$ for all values of y, x is always a positive number. Therefore, in the equation $y = \log_b x$, x is a positive number.

The logarithm of a negative number is not a real number.

The 1–1 property of exponential functions can be used to evaluate some logarithms.

Write equivalent exponential and logarithmic equations

EQUALITY OF EXPONENTS PROPERTY

For $b > 0$, $b \neq 1$, if $b^u = b^v$, then $u = v$.

EXAMPLES

1. If $3^x = 3^4$, then $x = 4$.

2. If $5^{3x} = 5^6$, then $3x = 6$.

Example 3

Evaluate: $\log_3\left(\frac{1}{9}\right)$

Solution:

$$\log_3\left(\frac{1}{9}\right) = x$$ Write an equation.

$$\frac{1}{9} = 3^x$$ Write the equation in its equivalent exponential form.

$$3^{-2} = 3^x$$ Write $\frac{1}{9}$ in exponential form using 3 as the base.

$$-2 = x$$ Solve for x using the Equality of Exponents Property.

$$\log_3\left(\frac{1}{9}\right) = -2$$

Write equivalent exponential and logarithmic equations

The Equality of Exponents Property can be used to solve some logarithmic equations.

Example 4

Solve $\log_6 x = 2$ for x.

Solution:

$$\log_6 x = 2$$

$$6^2 = x$$ Write $\log_6 x = 2$ in its equivalent exponential form.

$$36 = x$$

The solution is 36.

Write equivalent exponential and logarithmic equations

Logarithms base 10 are called **common logarithms**. We usually omit the base, 10, when writing the common logarithm of a number. Therefore, $\log_{10} x$ is written $\log x$.

To find the common logarithm of most numbers, a calculator is necessary. A calculator was used to find the value of log 384, shown below.

$$\log 384 \approx 2.584331224$$

Write equivalent exponential and logarithmic equations

When e (the base of the natural exponential function) is used as the base of a logarithm, $\log_e x$ is referred to as the **natural logarithm** and is abbreviated ln x.

This is read "el en x." The equivalent exponential form of $y = \ln x$ is $e^y = x$.

Using a calculator, we find that

$$\ln 23 \approx 3.135494216.$$

Example 5

Solve ln *x* = –1 for *x*. Round to the nearest ten-thousandth.

Solution:

$$\ln x = -1$$

$$e^{-1} = x \qquad \text{Use } \ln x = y \text{ is equivalent to } e^y = x.$$

$$0.3679 \approx x \qquad \text{Evaluate } e^{-1}.$$

The solution is 0.3679.

The properties of logarithms

The properties of logarithms

Because a logarithm is an exponent, the properties of logarithms are similar to the properties of exponents.

PRODUCT PROPERTY OF LOGARITHMS

For any positive real numbers x, y, and b, $b \neq 1$,

$$\log_b(xy) = \log_b x + \log_b y$$

EXAMPLES

1. $\log_7(9z) = \log_7 9 + \log_7 z$
2. $\log[(x - 2)(x + 3)] = \log(x - 2) + \log(x + 3)$
3. $\ln(xy) = \ln x + \ln y$

The properties of logarithms

The Product Property of Logarithms can be extended to more than two factors. For instance,

$$\log_b (xyz) = \log_b x + \log_b y + \log_b z$$

A second property of logarithms involves the logarithm of the quotient of two numbers.

This property of logarithms is also based on the fact that a logarithm is an exponent and that to divide two exponential expressions with the same base, we subtract the exponents.

The properties of logarithms

QUOTIENT PROPERTY OF LOGARITHMS

For any positive real numbers x, y, and b, $b \neq 1$, $\log_b \dfrac{x}{y} = \log_b x - \log_b y$.

EXAMPLES

1. $\log_3 \dfrac{y}{11} = \log_3 y - \log_3 11$
2. $\log \dfrac{x+1}{x-1} = \log(x+1) - \log(x-1)$
3. $\ln \dfrac{14}{w} = \ln 14 - \ln w$

A third property of logarithms, especially useful in computing the power of a number, is based on the fact that a logarithm is an exponent and that the power of an exponential expression is found by multiplying the exponents.

The properties of logarithms

POWER PROPERTY OF LOGARITHMS

For any positive real numbers x and b, $b \neq 1$, and for any real number r, $\log_b x^r = r \log_b x$.

EXAMPLES

1. $\log_4 x^5 = 5 \log_4 x$

2. $\log 3^{2x-1} = (2x - 1)\log 3$

3. $\ln \sqrt{x} = \ln x^{\frac{1}{2}} = \frac{1}{2} \ln x$

The properties of logarithms can be used in combination to write a logarithmic expression in **expanded form**.

Example 6

Write the logarithm in expanded form.

A. $\log_b(x^2\sqrt{y})$ **B.** $\ln \dfrac{x}{yz^2}$ **C.** $\log_8\sqrt{x^3y}$

Solution:

A. $\log_b(x^2\sqrt{y}) = \log_b x^2 + \log_b \sqrt{y}$ Use the Product Property of Logarithms.

$= \log_b x^2 + \log_b y^{\frac{1}{2}}$ Write $\sqrt{y} = y^{\frac{1}{2}}$.

$= 2\log_b x + \dfrac{1}{2}\log_b y$ Use the Power Property of Logarithms.

B. $\ln \dfrac{x}{yz^2} = \ln x - \ln(yz^2)$ Use the Quotient Property of Logarithms.

Example 6 – *Solution*

cont'd

$$= \ln x - (\ln y + \ln z^2)$$

Use the Product Property of Logarithms.

$$= \ln x - (\ln y + 2 \ln z)$$

Use the Power Property of Logarithms.

$$= \ln x - \ln y - 2 \ln z$$

Use the Distributive Property.

c. $\log_8 \sqrt{x^3 y} = \log_8 (x^3 y)^{\frac{1}{2}}$

Write the radical expression as an exponential expression.

$$= \frac{1}{2} \log_8 x^3 y$$

Use the Power Property of Logarithms.

$$= \frac{1}{2}(\log_8 x^3 + \log_8 y)$$

Use the Product Property of Logarithms.

Example 6 – *Solution*

cont'd

$$= \frac{1}{2}(3 \log_8 x + \log_8 y)$$

Use the Power Property of Logarithms.

$$= \frac{3}{2} \log_8 x + \frac{1}{2} \log_8 y$$

Use the Distributive Property.

The properties of logarithms

The properties of logarithms are also used to rewrite a logarithmic expression that is in expanded form as a single logarithm.

Example 7

Express as a single logarithm with a coefficient of 1.

A. $3 \log_5 x + \log_5 y - 2 \log_5 z$

B. $2(\log_4 x + 3 \log_4 y - 2 \log_4 z)$

C. $\frac{1}{3}(2 \ln x - 4 \ln y)$

Solution:

A. $3 \log_5 x + \log_5 y - 2 \log_5 z$

$= \log_5 x^3 + \log_5 y - \log_5 z^2$ Use the Power Property of Logarithms.

$= \log_5 x^3 y - \log_5 z^2$ Use the Product Property of Logarithms.

$= \log_5 \frac{x^3 y}{z^2}$ Use the Quotient Property of Logarithms.

Example 7 – *Solution*

cont'd

B. $2(\log_4 x + 3 \log_4 y - 2 \log_4 z)$

$= 2(\log_4 x + \log_4 y^3 - \log_4 z^2)$ Use the Power Property of Logarithms.

$= 2(\log_4 xy^3 - \log_4 z^2)$ Use the Product Property of Logarithms.

$= 2 \log_4 \dfrac{xy^3}{z^2}$ Use the Quotient Property of Logarithms.

$= \log_4 \left(\dfrac{xy^3}{z^2}\right)^2$ Use the Power Property of Logarithms.

$= \log_4 \dfrac{x^2y^6}{z^4}$ Simplify the power of the exponential expression.

Example 7 – *Solution*

cont'd

c. $\frac{1}{3}(2 \ln x - 4 \ln y)$

$= \frac{1}{3}(\ln x^2 - \ln y^4)$ Use the Power Property of Logarithms.

$= \frac{1}{3}\left(\ln \frac{x^2}{y^4}\right)$ Use the Quotient Property of Logarithms.

$= \ln\left(\frac{x^2}{y^4}\right)^{\frac{1}{3}}$ Use the Power Property of Logarithms. Write the exponential expression as a radical expression.

$= \ln\sqrt[3]{\frac{x^2}{y^4}}$

The properties of logarithms

There are three other properties of logarithms that are useful in simplifying logarithmic expressions.

OTHER PROPERTIES OF LOGARITHMS

Logarithmic Property of One

For any positive real number b, $b \neq 1$, $\log_b 1 = 0$.

EXAMPLES

1. $\log_7 1 = 0$ **2.** $\log 1 = 0$ **3.** $\ln 1 = 0$

Inverse Property of Logarithms

For any positive real numbers x and b, $b \neq 1$, $\log_b b^x = x$ and $b^{\log_b x} = x$.

EXAMPLES

1. $\log_5 5^x = x$ **2.** $\log 10^{3z-1} = 3z - 1$ **3.** $\ln e^{2x+1} = 2x + 1$

4. $8^{\log_8 x} = x$ **5.** $10^{\log(2y+3)} = 2y + 3$ **6.** $e^{\ln(3z-7)} = 3z - 7$

The properties of logarithms

1–1 Property of Logarithms

For any positive real numbers x, y, and b, $b \neq 1$, if $\log_b x = \log_b y$, then $x = y$.

EXAMPLES

1. If $\log_2(3x - 2) = \log_2(x + 4)$, then $3x - 2 = x + 4$.
2. If $\ln(x^2 + 1) = \ln(2x)$, then $x^2 + 1 = 2x$.

Although only common logarithms and natural logarithms are programmed into a calculator, the logarithms for other positive bases can be found.

CHANGE-OF-BASE FORMULA

$$\log_a N = \frac{\log_b N}{\log_b a}$$

Example 8

Evaluate $\log_7 32$. Round to the nearest ten-thousandth.

Solution:

$$\log_7 32 = \frac{\ln 32}{\ln 7}$$

Use the Change-of-Base Formula.
$N = 32, a = 7, b = e$

$$\approx 1.7810$$

Example 9

Rewrite $f(x) = -3 \log_7(2x - 5)$ in terms of natural logarithms.

Solution:

$$f(x) = -3 \log_7(2x - 5)$$

$$= -3 \cdot \frac{\ln(2x - 5)}{\ln 7}$$

Use the Change-of-Base Formula to rewrite $\log_7(2x - 5)$ as $\frac{\ln(2x - 5)}{\ln 7}$.

$$= -\frac{3}{\ln 7} \ln(2x - 5)$$