# Linear Functions and Inequalities in Two Variables

CHAPTER

3

# 3.2 Introduction to Functions

# Objectives

1. Evaluate a function
2. Graph of a function
3. Vertical line test

# Evaluate a function

# Evaluate a function

In mathematics, a *function* is used to describe a relationship between two quantities. Because two quantities are involved, it is natural to use ordered pairs.

**DEFINITION OF A FUNCTION**

A **function** is a set of ordered pairs in which no two ordered pairs have the same first coordinate. The **domain** of a function is the set of first coordinates of the ordered pairs; the **range** of a function is the set of second coordinates of the ordered pairs.

**EXAMPLES**

**1.** $\{(1, 2), (2, 4), (3, 6), (4, 8)\}$

Domain $= \{1, 2, 3, 4\}$

Range $= \{2, 4, 6, 8\}$

**2.** $\{(-1, 0), (0, 0), (1, 0), (2, 0), (3, 0)\}$

Domain $= \{-1, 0, 1, 2, 3\}$

Range $= \{0\}$

**3.** $\{(-5, 5), (-1, 1), (-7, 10), (4, 5)\}$

Domain $= \{-7, -5, -1, 4\}$

Range $= \{1, 5, 10\}$

# Evaluate a function

Now consider the set of ordered pairs {(1, 2), (4, 5), (7, 8), (4, 6)}. This set of ordered pairs is *not* a function. There are two ordered pairs, (4, 5) and (4, 6), with the same first coordinate. This set of ordered pairs is called a *relation*.

A **relation** is any set of ordered pairs. A function is a special type of relation. The concepts of domain and range apply to relations as well as to functions.

# Example 1

What are the domain and range of the following relation? Is the relation a function?
{(2, 4), (3, 6), (4, 8), (5, 10), (4, 6), (6, 10)}

Solution:

The domain is {2, 3, 4, 5, 6}.

The domain of the relation is the set of first coordinates of the ordered pairs.

The range is {4, 6, 8, 10}.

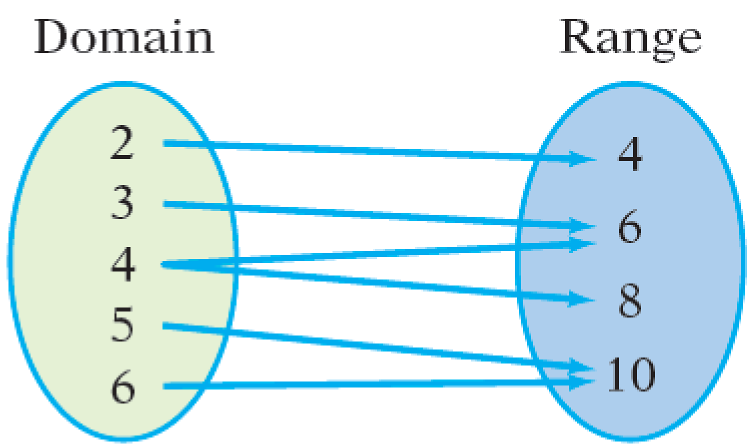
The range of the relation is the set of second coordinates of the ordered pairs.

The relation is not a function.

There are two ordered pairs with the same first coordinate and different second coordinates, (4, 8) and (4, 6).

# Evaluate a function

For each element in the domain of a relation, there is a corresponding element in the range of the relation. A possible diagram of the relation in Example 1 is




$\{(2, 4), (3, 6), (4, 8), (5, 10), (4, 6), (6, 10)\}$

Note from the diagram that there are two arrows from 4 in the domain to different elements, 6 and 8, in the range.

# Evaluate a function

This means that there are two ordered pairs, (4, 6) and (4, 8), with the same first coordinate and different second coordinates. The relation is not a function.

Although a function can be described in terms of ordered pairs, in a table, or by a graph, a major focus of this text will be functions represented by equations in two variables.

# Evaluate a function

For instance, the equation $s = 16t^2$ describes the relationship between the time $t$, a pilot fell from a helium-filled-balloon and the distance $s$, he fell. The ordered pairs can be written as $(t, s)$, where $s = 16t^2$.

By substituting $16t^2$ for $s$, we can also write the ordered pairs as $(t, 16t^2)$. Because the distance he falls *depends* on how long he has been falling, $s$ is called the **dependent variable** and $t$ is called the **independent variable**.

For the equation $s = 16t^2$, we say that "distance is a function of time."

# Evaluate a function

A function can be thought of as a rule that pairs one number with another number. For instance, the **square function** pairs a real number with its square.

This function can be represented by the equation $y = x^2$. This equation states that, given any element $x$ in the domain, the value of $y$ in the range is the square of $x$.

# Evaluate a function

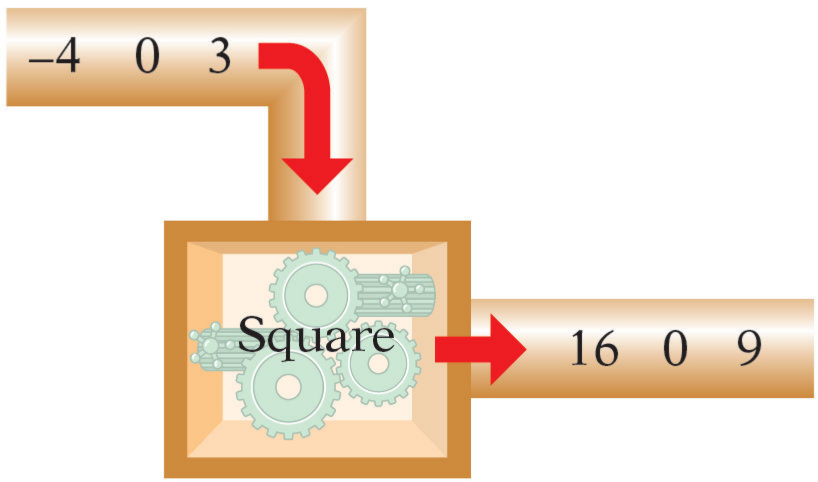
In this case, the independent variable is *x* and the dependent variable is *y*.

When $x = 6$, $y = 36$, and one ordered pair of the function is (6, 36). When $x = –7$, $y = 49$.

The ordered pair (–7, 49) belongs to the function. Other ordered pairs of the function are (–2, 4), $\left(\frac{3}{4}, \frac{9}{16}\right)$, and (–1.1, 1.21).

# Evaluate a function

In many cases, you may think of a function as a machine that turns one number into another number. For instance, you can think of the square function machine at the right taking an **input** (an element from the domain) and creating an **output** (an element of the range) that is the square of the input.




$f(x) = x^2$

# Evaluate a function

**Function notation** is frequently used for those equations that define functions. Just as $x$ is commonly used as a variable for a number, the letter $f$ is commonly used to name a function. For instance, using the square function, we can write $f(x) = x^2$.

The symbol $f(x)$ is read "the value of $f$ at $x$" or " $f$ of $x$." The symbol $f(x)$ is the **value of the function** and represents the value of the dependent variable for a given value of the independent variable.

# Evaluate a function

This is the value of the function.
It is the value of the dependent variable.

$$f(x) = x^2$$

The name of the function is $f$.

This is an algebraic expression that defines the relationship between the dependent variable and the independent variable.

The process of finding the value of $f(x)$ for a given value of $x$ is called **evaluating the function**.

# Evaluate a function

For instance, to evaluate $f(x) = x^2$ when $x$ is 4, replace $x$ by 4 and simplify.

$$\begin{aligned} f(x) &= x^2 \\ f(4) &= 4^2 \\ &= 16 \end{aligned}$$

**Replace $x$ by 4. Then simplify.**

The *value* of the function is 16 when $x = 4$. An ordered pair of the function is (4, 16).

We write $y = f(x)$ to emphasize the relationship between the independent variable $x$ and the dependent variable $y$.

**Remember:** $y$ and $f(x)$ are different symbols for the same number. Also, the *name* of the function is $f$; the *value* of the function at $x$ is $f(x)$.

# Example 2

Let $q(r) = 2r^3 + 5r^2 - 6$.

**A.** Find $q(-3)$.

**B.** Find the value of $q(r)$ when $r = 2$.

Solution:

**A.** $q(r) = 2r^3 + 5r^2 - 6$

$q(-3) = 2(-3)^3 + 5(-3)^2 - 6$ **Replace *r* by −3.**

$= 2(-27) + 5(9) - 6$

$= -54 + 45 - 6$ **Simplify.**

$q(-3) = -15$

# Example 2

**B.** To find the value of $q(r)$ when $r = 2$ means to evaluate the function when $r = 2$.

$$q(r) = 2r^3 + 5r^2 - 6$$

$$q(2) = 2(2)^3 + 5(2)^2 - 6$$ Replace $r$ by **2**.

$$= 2(8) + 5(4) - 6$$

$$= 16 + 20 - 6$$ Simplify.

$$q(2) = 30$$

The value of the function when $r = 2$ is 30.

# Evaluate a function

The range of a function contains all the elements that result from evaluating the function for each element of the domain. If the domain contains an infinite number of elements, as in the case of the falling pilot, it may be difficult to find the range.

However, if the domain has only a finite number of elements, then the range can be found by evaluating the function for each element of the domain.

# Example 4

Find the range of $f(x) = x^2 + 2x - 1$ if the domain is $\{-3, -2, -1, 0, 1\}$.

Solution:

Evaluate the function for each element of the domain. The range includes the values of $f(-3)$, $f(-2)$, $f(-1)$, $f(0)$, and $f(1)$.

$$f(x) = x^2 + 2x - 1$$

$$f(-3) = (-3)^2 + 2(-3) - 1 = 2$$

$$f(-2) = (-2)^2 + 2(-2) - 1 = -1$$

$$f(-1) = (-1)^2 + 2(-1) - 1 = -2$$

# Example 4 – *Solution*

cont'd

$$f(0) = (0)^2 + 2(0) - 1 = -1$$

$$f(1) = (1)^2 + 2(1) - 1 = 2$$

This range is {–2, –1, 2}.

# Example 5

Given $f(x) = 3x + 1$, find a number $c$ in the domain of $f$ such that $f(c) = 7$. Write the corresponding ordered pair of the function.

Solution:

$$f(x) = 3x + 1$$

$$f(c) = 3c + 1 \qquad \text{Replace } x \text{ by } c.$$

$$7 = 3c + 1 \qquad f(c) = 7$$

$$6 = 3c \qquad \text{Solve for } c.$$

$$2 = c$$

The value of $c$ is 2. The corresponding ordered pair is (2, 7).

# Graph of a function

# Graph of a function

The **graph of a function** is the graph of the ordered pairs that belong to the function. Graphing a function is similar to graphing an equation in two variables.

First, evaluate the function at selected values of *x* and plot the corresponding ordered pairs. Then connect the points with a smooth curve to form the graph.

# Example 6

Graph $h(x) = x^2 – 3$. First evaluate the function when $x = –3, –2, –1, 0, 1, 2$, and 3. Plot the resulting ordered pairs. Then connect the points to form the graph.

Solution:

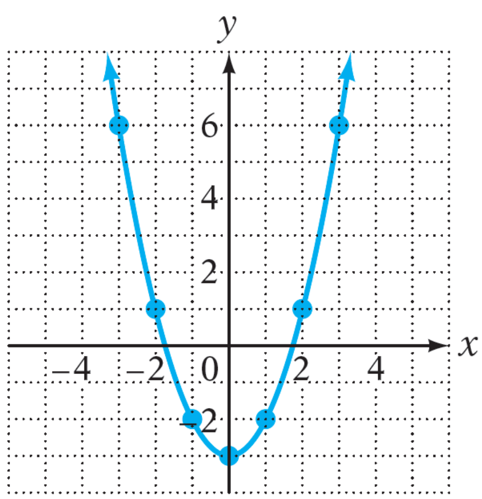
Evaluate the function for the given values of $x$. This will produce some ordered pairs of the function. The results can be recorded in a table.

# Example 6 – *Solution*

cont'd

Plot the ordered pairs and then connect the points to form the graph.

| $x$ | $y = h(x) = x^2 - 3$ | $y$ | $(x, y)$ |
|---|---|---|---|
| $-3$ | $h(-3) = (-3)^2 - 3 = 6$ | $6$ | $(-3, 6)$ |
| $-2$ | $h(-2) = (-2)^2 - 3 = 1$ | $1$ | $(-2, 1)$ |
| $-1$ | $h(-1) = (-1)^2 - 3 = -2$ | $-2$ | $(-1, -2)$ |
| $0$ | $h(0) = (0)^2 - 3 = -3$ | $-3$ | $(0, -3)$ |
| $1$ | $h(1) = (1)^2 - 3 = -2$ | $-2$ | $(1, -2)$ |
| $2$ | $h(2) = (2)^2 - 3 = 1$ | $1$ | $(2, 1)$ |
| $3$ | $h(3) = (3)^2 - 3 = 6$ | $6$ | $(3, 6)$ |

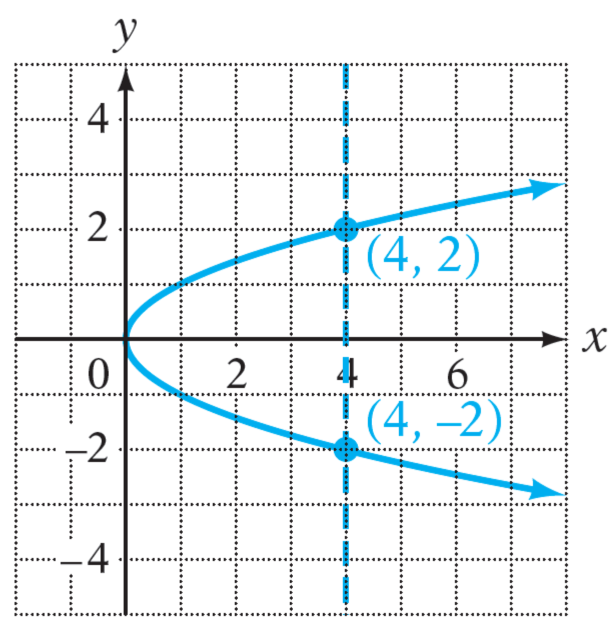


Vertical line test

# Vertical line test

A graph is a visual description of a relation. For the relation graphed below, the two ordered pairs (4, 2) and (4, –2) belong to the graph, and those points lie on a vertical line.

Because there are two ordered pairs with the same first coordinate, the set of ordered pairs of the graph is not a function. With this observation in mind, we can make use of a quick method to determine whether a graph is the graph of a function.



# Vertical line test

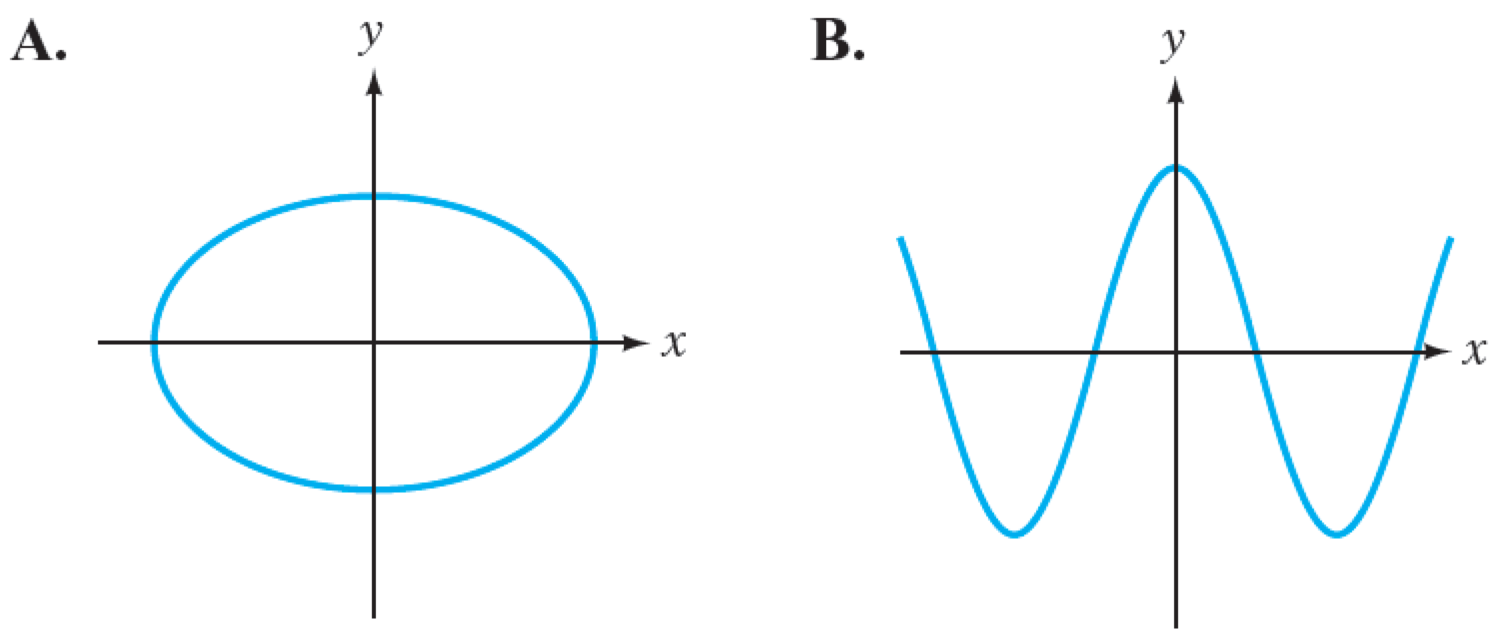
Previous graphical interpretation of a function is often described by saying that each value in the domain of the function is paired with *exactly one* value in the range of the function.

**VERTICAL LINE TEST FOR THE GRAPH OF A FUNCTION**

If every vertical line intersects a graph at most once, then the graph is the graph of a function.

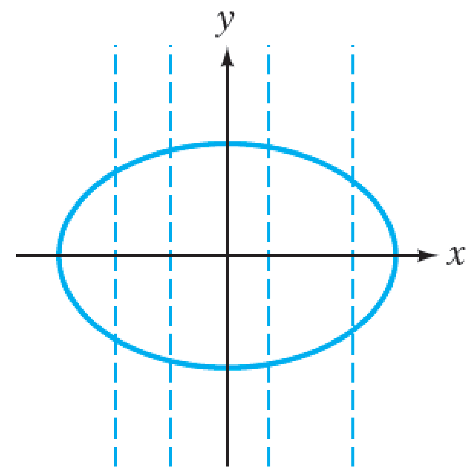
# Example 8

Use the vertical line test to determine whether the graph is the graph of a function.



# Example 8 – *Solution*

**A.** As shown at the right, there are vertical lines that intersect the graph at more than one point. Therefore, the graph is not the graph of a function.




**B.** For the graph at the right, every vertical line intersects the graph at most once. Therefore, the graph is the graph of a function.