CHAPTER 2

Solving Equations and Inequalities

2.7 Inequalities in One Variable

Objectives

1. Interval notation and set operations
2. Solve inequalities in one variable
3. Solve compound inequalities
4. Application problems

Interval notation and set operations

Interval notation and set operations

The **roster method** of writing a set encloses a list of the elements of the set in braces. We used this method at the beginning of this section to define sets of numbers.

Using the roster method, the set of even natural numbers less than 10 is written {2, 4, 6, 8}. This is an example of a **finite set**; all the elements can be listed. The set of natural numbers, {1, 2, 3, 4,…}, is an **infinite set**; it is impossible to list all the elements.

The **empty set**, or **null set**, is the set that contains no elements. The symbol Ø or { } is used to represent the empty set.

Example 1

Use the roster method to write the set of natural numbers less than 10.

Solution:

{1, 2, 3, 4, 5, 6, 7, 8, 9}

Interval notation and set operations

A second method of representing a set is **set-builder notation**. Set-builder notation can be used to describe almost any set, but it is especially useful when writing infinite sets.

In set-builder notation, the set of integers greater than –3 is written

$$\{x \mid x > -3,\ x \in \text{integers}\}$$

and is read "the set of all *x* such that *x* is greater than –3 and *x* is an element of the integers." This is an infinite set.

It is impossible to list all the elements of the set, but the set can be described by using set-builder notation.

Example 2

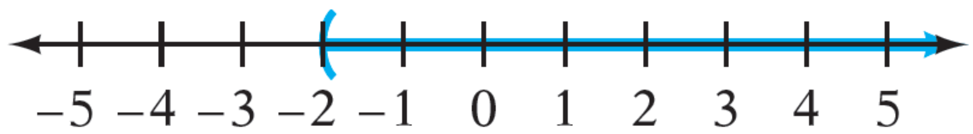
Use set-builder notation to write the set of real numbers greater than –2.

Solution:

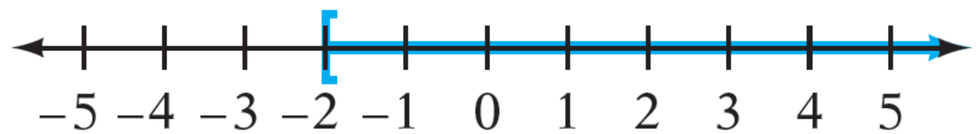
$$\{x \mid x > -2\}$$

Interval notation and set operations

The graph of a set of real numbers written in set-builder notation can be shown on a number line. The graph of $\{x|x > -2\}$ is shown below. The parenthesis on the graph indicates that –2 is not part of the set.



The graph of $\{x|x \geq -2\}$ is shown below. The bracket on the graph indicates that –2 is included in the set.

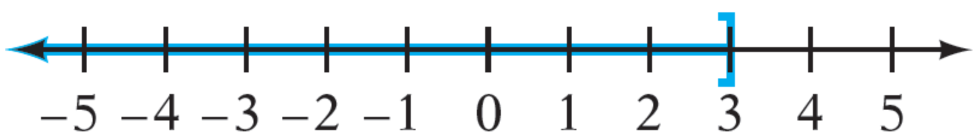


Example 3

Graph: $\{x | x \leq 3\}$

Solution:

The set is the real numbers less than or equal to 3.



Draw a right bracket at 3, and darken the number line to the left of 3.

Interval notation and set operations

It is also possible to graph a set of real numbers *between* two given numbers.

Given two real numbers, an **interval** is the set of all real numbers between the given numbers. The two numbers are the **endpoints** of the interval.

For example, the set $\{x \mid -1 < x < 3\}$ represents the interval of all real numbers between –1 and 3. The endpoints of this interval are –1 and 3.

A **closed interval** includes both endpoints; an **open interval** contains neither endpoint; a **half-open interval** contains one endpoint but not the other.

Interval notation and set operations

For example, the set $\{x \mid -1 < x < 3\}$ is an open interval.

Intervals can be represented in set-builder notation or **interval notation**. In interval notation, the brackets or parentheses that are used to graph the set are written with the endpoints of the interval.

The set $\{x \mid 0 \leq x < 4\}$ shown above is written $[0, 4)$ in interval notation; 0 and 4 are the endpoints.

Interval notation and set operations

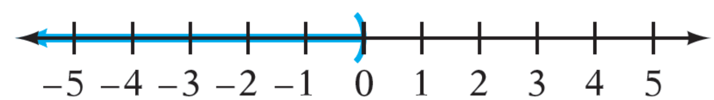
To indicate an interval that extends forever in one or both directions using interval notation, we use the **infinity symbol** ∞ or the **negative infinity symbol** $-\infty$.

The infinity symbol is not a number; it is simply a notation used to indicate that the interval is unlimited.

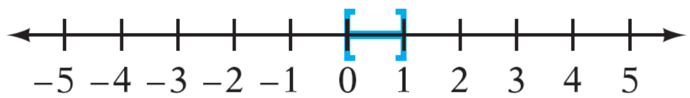
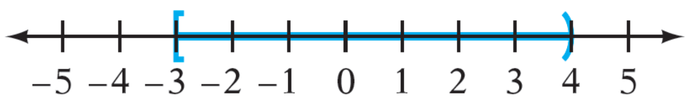
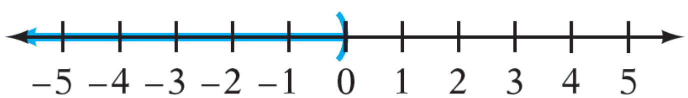
In interval notation, a parenthesis is always used to the right of an infinity symbol or to the left of a negative infinity symbol.

Example 4

Use the given notation or graph to supply the notation or graph that is marked with a question mark.

	Set-builder notation	Interval notation	Graph
A.	$\{x \mid 0 \leq x \leq 1\}$	?	?
B.	?	$[-3, 4)$	?
C.	?	?	

Example 4 – *Solution*

	Set-builder notation	Interval notation	Graph
A.	$\{x \mid 0 \le x \le 1\}$	$[0, 1]$	
B.	$\{x \mid -3 \le x < 4\}$	$[-3, 4)$	
C.	$\{x \mid x < 0\}$	$(-\infty, 0)$	

Interval notation and set operations

Just as operations such as addition and multiplication are performed on real numbers, operations are performed on sets. Two operations performed on sets are *union* and *intersection*.

UNION OF TWO SETS

The **union** of two sets, written $A \cup B$, is the set of all elements that belong to either A **or** B. In set-builder notation, this is written

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

EXAMPLES

1. Given $A = \{2, 3, 4, 5, 6\}$ and $B = \{4, 5, 6, 7, 8\}$, $A \cup B = \{2, 3, 4, 5, 6, 7, 8\}$. Note that the elements 4, 5, and 6, which belong to both sets, are listed only once.
2. Given $C = \{-3, -1, 1, 3\}$ and $D = \{-2, 0, 2\}$, $C \cup D = \{-3, -2, -1, 0, 1, 2, 3\}$.
3. Given $X = \{0, 2, 4, 6, 8\}$ and $Y = \{4, 8\}$, $X \cup Y = \{0, 2, 4, 6, 8\}$.

Interval notation and set operations

INTERSECTION OF TWO SETS

The **intersection** of two sets, written $A \cap B$, is the set of all elements that are common to both A **and** B. In set-builder notation, this is written

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

EXAMPLES

1. Given $A = \{2, 3, 4, 5, 6\}$ and $B = \{4, 5, 6, 7, 8\}$, $A \cap B = \{4, 5, 6\}$.

2. Given $C = \{-3, -1, 1, 3\}$ and $D = \{-2, 0, 2\}$, $C \cap D = \varnothing$. There are no elements common to both C and D.

3. Given $X = \{0, 2, 4, 6, 8\}$ and $Y = \{4, 8\}$, $X \cap Y = \{4, 8\}$.

Set operations also can be performed on intervals.

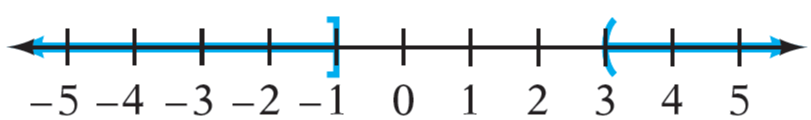
Example 5

Graph: **A.** $\{x|x \leq -1\} \cup \{x|x > 3\}$ **B.** $(-\infty, 3) \cap [-1, \infty)$

Solution:

A. The set $\{x|x \leq -1\} \cup \{x|x > 3\}$ is the set of real numbers less than or equal to –1 *or* greater than 3.

This set can be written $\{x|x \leq -1 \text{ or } x > 3\}$.



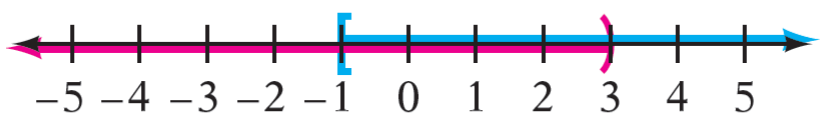
The graph of $\{x|x \leq -1$ or $x > 3\}$ contains all the points on the graphs of $x \leq -1$ and $x > 3$.

Example 5 – *Solution*

cont'd

B. The set $(-\infty, 3) \cap [-1, \infty)$ set of real numbers less than 3 *and* greater than or equal to –1.

The graph of $(-\infty, 3)$ is shown in red, and the graph of $[-1, \infty)$ is shown in blue.

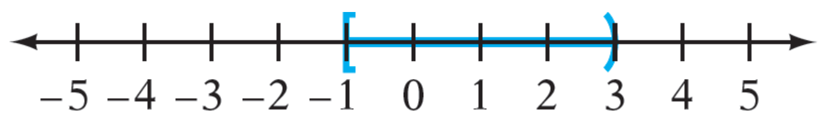


Real numbers that are elements of both $(-\infty, 3)$ *and* $[-1, \infty)$ correspond to the points in the section of overlap; thus, $(-\infty, 3) \cap [-1, \infty) = [-1, 3)$.

Example 5 – *Solution*

cont'd

Note that 3 is not an element of $(-\infty, 3)$. Therefore, 3 is not an element of the intersection of the sets.



Solve inequalities in one variable

Solve inequalities in one variable

The **solution set of an inequality** is a set of numbers, each element of which, when substituted for the variable, results in a true Inequality.

The inequality at the right is true if the variable is replaced by 3, for example, or by –1.98, or by $\frac{2}{3}$.

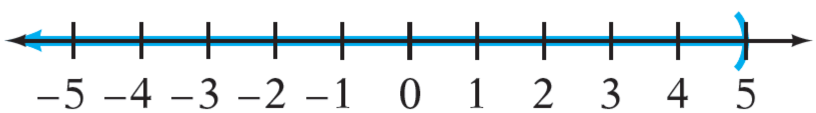
$$\begin{aligned} x - 1 &< 4 \\ 3 - 1 &< 4 \\ -1.98 - 1 &< 4 \\ \frac{2}{3} - 1 &< 4 \end{aligned}$$

There are many values of the variable *x* that will make the inequality $x - 1 < 4$ true.

Solve inequalities in one variable

The solution set of the inequality is any number less than 5. The solution set can be written in set-builder notation as $\{x \mid x < 5\}$ or in interval notation as $(-\infty, 5)$.

The graph of the solution set of $x - 1 < 4$ is shown below.



In solving an inequality, use the Addition and Multiplication Properties of Inequalities to rewrite the inequality in the form *variable* < *constant* or *variable* > *constant*.

Solve inequalities in one variable

THE ADDITION PROPERTY OF INEQUALITIES

If $a > b$ and c is a real number, then the inequalities $a > b$ and $a + c > b + c$ have the same solution set.

If $a < b$ and c is a real number, then the inequalities $a < b$ and $a + c < b + c$ have the same solution set.

EXAMPLES

1.	Begin with a true inequality.	$6 > 2$
	Add -8 to each side.	$6 + (-8) > 2 + (-8)$
	Simplify. The inequality is true.	$-2 > -6$
2.	Begin with a true inequality.	$-9 < -3$
	Add 5 to each side.	$-9 + 5 < -3 + 5$
	Simplify. The inequality is true.	$-4 < 2$

Solve inequalities in one variable

The Addition Property of Inequalities states that the same number can be added to each side of an inequality without changing the solution set of the inequality.

This property is also true for an inequality that contains the symbol $\leq$ or $\geq$.

The Addition Property of Inequalities is used to remove a term from one side of an inequality by adding the additive inverse of that term to each side of the inequality.

Solve inequalities in one variable

Because subtraction is defined in terms of addition, the same number can be subtracted from each side of an inequality without changing the solution set of the inequality.

Solve inequalities in one variable

THE MULTIPLICATION PROPERTY OF INEQUALITIES

RULE 1

If $a > b$ and $c > 0$, then the inequalities $a > b$ and $ac > bc$ have the same solution set.

If $a < b$ and $c > 0$, then the inequalities $a < b$ and $ac < bc$ have the same solution set.

EXAMPLES

1. Begin with a true inequality. $5 > 2$

Multiply each side of the inequality by *positive* 3. $3(5) > 3(2)$

Simplify. The inequality is true. $15 > 6$

2. Begin with a true inequality. $-5 < -3$

Multiply each side of the inequality by *positive* 4. $4(-5) < 4(-3)$

Simplify. The inequality is true. $-20 < -12$

Solve inequalities in one variable

RULE 2

If $a > b$ and $c < 0$, then the inequalities $a > b$ and $ac < bc$ have the same solution set.

If $a < b$ and $c < 0$, then the inequalities $a < b$ and $ac > bc$ have the same solution set.

EXAMPLES

3. Begin with a true inequality. $9 > 3$

Multiply each side of the inequality by *negative* 2 and reverse the inequality symbol. $(-2)9 < (-2)3$

Simplify. The inequality is true. $-18 < -6$

4. Begin with a true inequality. $-6 < -4$

Multiply each side of the inequality by *negative* 3 and reverse the inequality symbol. $(-3)(-6) > (-3)(-4)$

Simplify. The inequality is true. $18 > 12$

Solve inequalities in one variable

The Multiplication Property of Inequalities is also true for the symbol $\leq$ or $\geq$.

Rule 1 states that when each side of an inequality is multiplied by a positive number, the inequality symbol remains the same.

However, Rule 2 states that when each side of an inequality is multiplied by a negative number, the inequality symbol must be reversed.

Solve inequalities in one variable

Because division is defined in terms of multiplication, when each side of an inequality is divided by a positive number, the inequality symbol remains the same.

When each side of an inequality is divided by a negative number, the inequality symbol must be reversed.

The Multiplication Property of Inequalities is used to remove a coefficient from one side of an inequality.

Example 6

Solve: $x + 3 > 4x + 6$

Write the solution set in set-builder notation.

Solution:

$$x + 3 > 4x + 6$$

$$-3x + 3 > 6$$ **Subtract $4x$ from each side of the inequality.**

$$-3x > 3$$ **Subtract 3 from each side of the inequality.**

$$x < -1$$ **Divide each side of the inequality by -3, and reverse the inequality symbol.**

The solution set is $\{x | x < -1\}$. **Write the solution set in set-builder notation.**

Solve inequalities in one variable

When an inequality contains parentheses, the first step in solving the inequality is to use the Distributive Property to remove the parentheses.

Example 7

Solve: $5(x - 2) \geq 9x - 3(2x - 4)$

Write the solution set in set-builder notation.

Solution:

$$5(x - 2) \geq 9x - 3(2x - 4)$$

$$5x - 10 \geq 9x - 6x + 12$$

Use the Distributive Property to remove parentheses.

$$5x - 10 \geq 3x + 12$$

Simplify.

$$2x - 10 \geq 12$$

Subtract $3x$ from each side of the inequality.

Example 7 – *Solution*

cont'd

$2x \geq 22$	Add 10 to each side of the inequality.
$x \geq 11$	Divide each side of the inequality by 2.
The solution set is $\{x \mid x \geq 11\}$.	Write the solution set in set-builder notation.

Solve compound inequalities

Solve compound inequalities

A **compound inequality** is formed by joining two inequalities with a connective word such as “and” or “or.” The inequalities shown below are compound inequalities.

$$2x < 4 \quad \text{and} \quad 3x - 2 > -8$$
$$2x + 3 > 5 \quad \text{or} \quad x + 2 < 5$$

The solution set of a compound inequality with the connective word *or* is the *union* of the solution sets of the two inequalities.

Example 8

Solve: $3 - 4x > 7$ or $4x + 5 < 9$

Write the solution set in interval notation.

Solution:

$$3 - 4x > 7 \quad \text{or} \quad 4x + 5 < 9$$

Solve each inequality.

$$-4x > 4 \qquad\qquad 4x < 4$$

$$x < -1 \qquad\qquad x < 1$$

$$(-\infty, -1) \qquad (-\infty, 1)$$

$$(-\infty, -1) \cup (-\infty, 1) = (-\infty, 1)$$

Find the union of the solution sets.

The solution set is $(-\infty, 1)$

Solve compound inequalities

The solution set of a compound inequality with the connective word *and* is the set of all elements common to the solution sets of both inequalities.

Therefore, it is the *intersection* of the solution sets of the two inequalities.

Example 9

Solve: $11 - 2x > -3 \text{ and } 7 - 3x < 4$

Write the solution set in set-builder notation.

Solution:

$$11 - 2x > -3 \quad \text{and} \quad 7 - 3x < 4$$

Solve each inequality.

$$-2x > -14 \qquad -3x < -3$$

$$x < 7 \qquad x > 1$$

$$\{x|x < 7\} \qquad \{x|x > 1\}$$

$$\{x|x < 7\} \cap \{x|x > 1\} = \{x|1 < x < 7\}$$

Find the intersection of the solution sets.

The solution set is $\{x|1 < x < 7\}$.

Solve compound inequalities

Some inequalities that use the connective word *and* can be written using a more compact notation.

For instance, the inequality $2x + 1 > -3$ *and* $2x + 1 \leq 7$ can be written as $-3 < 2x + 1 \leq 7$.

When the compound inequality is written in this form, an alternative method of solving the compound inequality can be used.

Example 10

Solve: $1 < 3x - 5 < 4$

Write the solution set in interval notation.

Solution:

$$1 < 3x - 5 < 4$$

$$1 + 5 < 3x - 5 + 5 < 4 + 5$$

Add 5 to each of the three parts of the inequality.

$$6 < 3x < 9$$

Simplify.

$$\frac{6}{3} < \frac{3x}{3} < \frac{9}{3}$$

Divide each of the three parts of the inequality by 3.

$$2 < x < 3$$

The solution set is $(2, 3)$.

Write the solution set in interval notation.

Application problems

Example 11

An average score of 80 to 89 in a history course receives a B grade. A student has grades of 72, 94, 83, and 70 on four exams. Find the range of scores on the fifth exam that will give the student a B for the course.

Strategy:

To find the range of scores, write and solve an inequality using S to represent the score on the fifth exam.

Example 11 – *Solution*

$$\frac{72 + 94 + 83 + 70 + S}{5}$$

The student's average score is the sum of the five scores, divided by 5.

$$80 \le \frac{72 + 94 + 83 + 70 + S}{5} \le 89$$

Write the inequality that puts the student's average score between 80 and 89, inclusive.

$$80 \le \frac{319 + S}{5} \le 89$$

Simplify the numerator.

$$5(80) \le 5\left(\frac{319 + S}{5}\right) \le 5(89)$$

Multiply each part of the inequality by 5.

Example 11 – *Solution*

cont'd

$$400 \leq 319 + S \leq 445$$

$$400 - 319 \leq 319 - 319 + S \leq 445 - 319$$ **Subtract 319 from each part of the inequality.**

$$81 \leq S \leq 126$$

The maximum score on an exam is 100, so eliminate values of *S* above 100. The range of scores that will give the student a B for the course is $81 \leq S \leq 100$.