Chapter 9 Surfaces and Solids

9.4 Polyhedrons and Spheres

POLYHEDRONS

Polyhedrons

When two planes intersect, the angle formed by two half-planes with a common edge (the line of intersection) is a **dihedral angle**.

The angle shown in Figure 9.38 is such an angle.

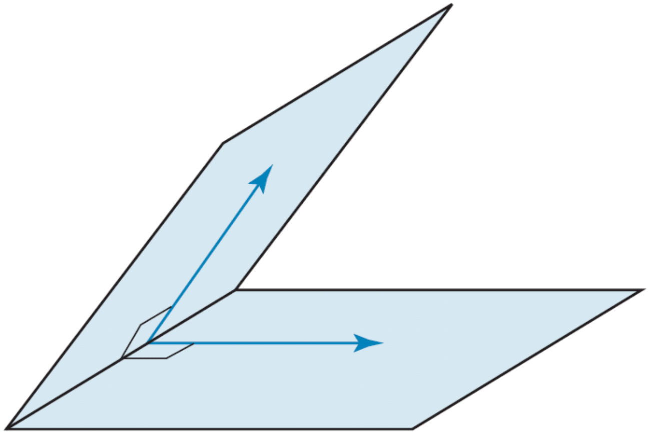


Figure 9.38

Polyhedrons

In Figure 9.38, the measure of the dihedral angle is the same as that of the angle determined by two rays that

1. have a vertex (the common endpoint) on the edge.

2. lie in the planes so that they are perpendicular to the edge.

Polyhedrons

A **polyhedron** (plural *polyhedrons or polyhedra*) is a solid bounded by four or more plane regions.

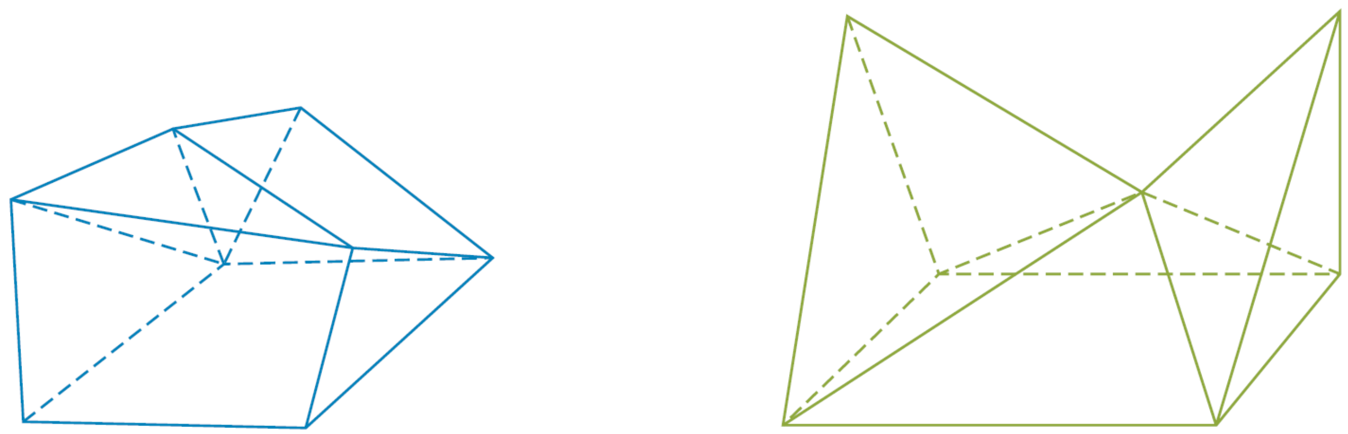
Polygons form the **faces** of the solid, and the segments common to these polygons are the **edges** of the polyhedron.

Endpoints of the edges are the **vertices** of the polyhedron.

When a polyhedron is **convex**, each face determines a plane for which all remaining faces lie on the same side of that plane.

Polyhedrons

Figure 9.39(a) illustrates a convex polyhedron, and Figure 9.39(b) illustrates a concave polyhedron. In the concave polyhedron, at least one diagonal lies in the exterior of the polyhedron.



Convex polyhedron (a)

Concave polyhedron (b)

Figure 9.39

Polyhedrons

The prisms and pyramids were special types of polyhedrons.

For instance, a pentagonal pyramid can be described as a hexahedron because it has six faces.

Because some of their surfaces do not lie in planes, the cylinders and cones are not polyhedrons.

Polyhedrons

Leonhard Euler (Swiss, 1707–1763) found that the number of vertices, edges, and faces of any polyhedron are related by **Euler's equation**.

This equation is given in the following theorem, which is stated without proof.

Theorem 9.4.1 (Euler's Equation)

The number of vertices V, the number of edges E, and the number of faces F of a polyhedron are related by the equation

$$V + F = E + 2$$

Example 1

Verify Euler's equation for the

(a) tetrahedron and

(b) square pyramid shown in Figure 9.40(a) and (b), respectively.

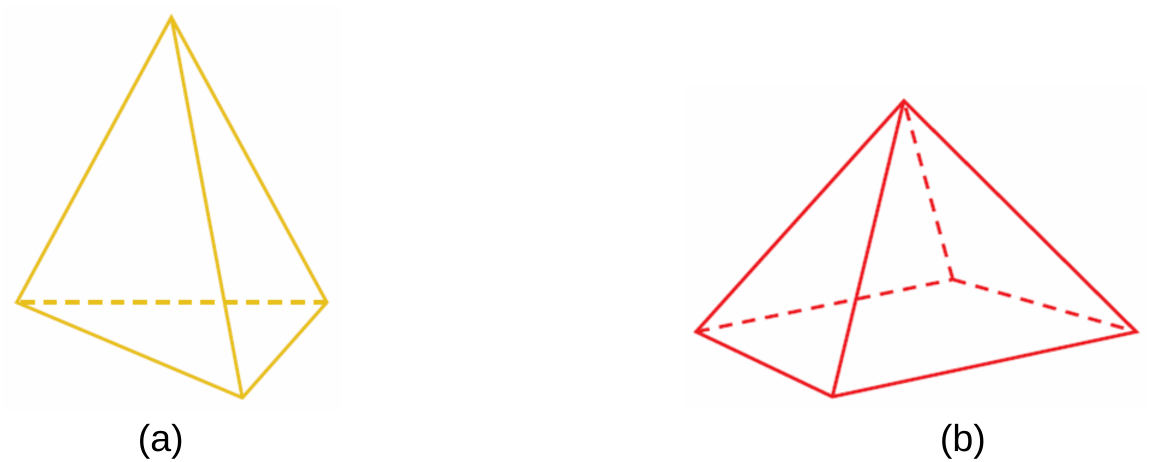


Figure 9.40

Example 1 – *Solution*

a) The tetrahedron has four vertices ($V = 4$), six edges ($E = 6$), and four faces ($F = 4$).

So Euler's equation becomes 4 + 4 = 6 + 2, which is true.

b) The pyramid has five vertices (apex + vertices from the base), eight edges (4 base edges + 4 lateral edges), and five faces (4 triangular faces + 1 square base).

Now $V + F = E + 2$ becomes 5 + 5 = 8 + 2, which is also true.

REGULAR POLYHEDRONS

Regular Polyhedrons

Definition

A **regular polyhedron** is a convex polyhedron whose faces are congruent regular polygons, all of the same type.

There are exactly five regular polyhedrons, named as follows:

1. Regular **tetrahedron**: with 4 faces that are congruent equilateral triangles

2. Regular **hexahedron** (or **cube**): with 6 faces that are congruent squares

Regular Polyhedrons

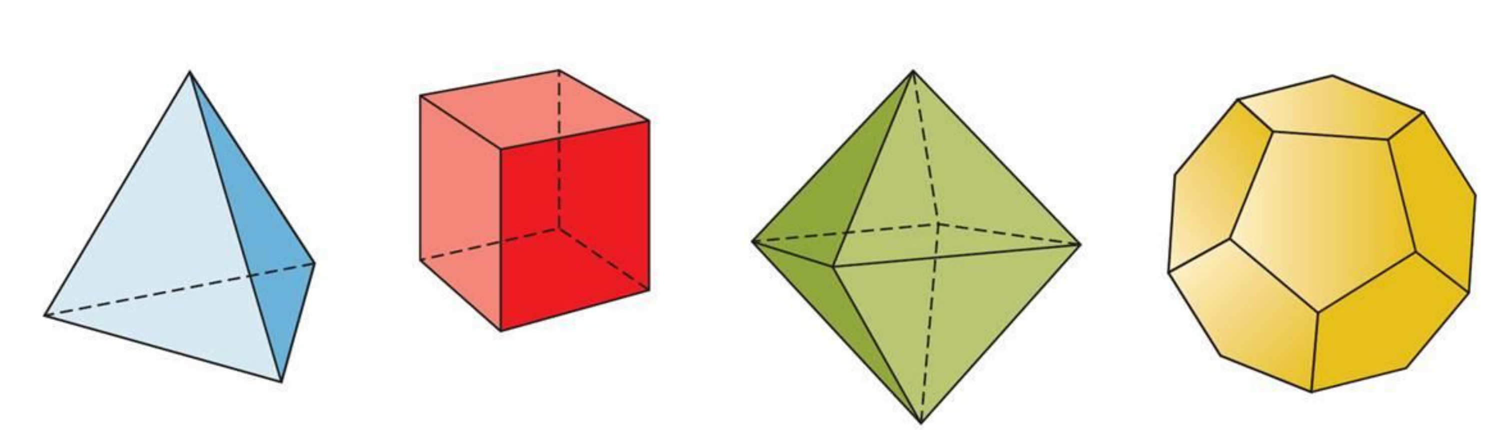
3. Regular **octahedron**: with 8 faces that are congruent equilateral triangles

4. Regular **dodecahedron**: with 12 faces that are congruent regular pentagons

5. Regular **icosahedron**: with 20 faces that are congruent equilateral triangles

Regular Polyhedrons

Four of the regular polyhedrons are shown in Figure 9.41.



Tetrahedron Hexahedron Octahedron Dodecahedron

Regular Polyhedrons

Figure 9.41

Regular Polyhedrons

Because each regular polyhedron has a central point, each solid is said to have a center.

Except for the tetrahedron, these polyhedrons have *point symmetry* at the center.

All regular polyhedra have line symmetry and plane symmetry as well.

Example 2

Consider a die that is a regular tetrahedron with faces numbered 1, 2, 3, and 4. Assuming that each face has an equal chance of being rolled, what is the likelihood (probability) that one roll produces (a) a “1”? (b) a result larger than “1”?

Solution:

a) With four equally likely results (1, 2, 3, and 4), the probability of a “1” is $\frac{1}{4}$.

Example 2 – *Solution*

cont'd

b) With four equally likely results (1, 2, 3, and 4) and three "favorable" outcomes (2, 3, and 4), the probability of rolling a number larger than a "1" is $\frac{3}{4}$.

SPHERES

Spheres

Another type of solid with which you are familiar is the sphere. Although the surface of a basketball correctly depicts the sphere, we often use the term *sphere* to refer to a solid like a baseball as well.

A sphere can be inscribed in or circumscribed about any regular polyhedron because it has point symmetry about its center.

In space, the sphere is characterized in three ways:

1. A **sphere** is the locus of points at a fixed distance r from a given point O. Point O is known as the **center** of the sphere, even though it is not a part of the spherical surface.
2. A **sphere** is the surface determined when a circle (or semicircle) is rotated about any of its diameters.
3. A **sphere** is the surface that represents the theoretical limit of an "inscribed" regular polyhedron whose number of faces increases without limit.

Spheres

Each characterization of the sphere has its advantages.

Characterization 1

In Figure 9.42, a sphere was generated as the locus of points in space at a distance *r* from point *O*. The line segment $\overline{OP}$ is a **radius** of sphere *O*, and $\overline{QP}$ is a **diameter** of the sphere. For the earth, the equator is a great circle that separates the earth into two **hemispheres**.

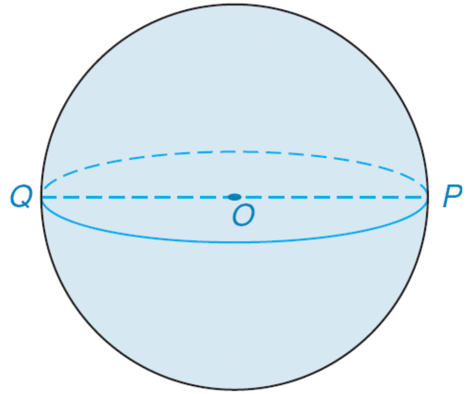


Figure 9.42

SURFACE AREA OF A SPHERE

Surface Area of a Sphere

Characterization 2

The following theorem claims that the surface area of a sphere equals four times the area of a great circle of that sphere. This theorem, treats the sphere as a surface of revolution.

Theorem 9.4.2

The surface area S of a sphere whose radius has length r is given by $S = 4\pi r^2$.

Example 3

Find the surface area of a sphere whose radius is $r = 7$ in. Use your calculator to approximate the result.

Solution:

$$S = 4\pi r^2 \rightarrow S = 4\pi \cdot 7^2 = 196\pi \text{ in}^2$$

Then $S \approx 615.75 \text{ in}^2$.

Surface Area of a Sphere

Although half of a circle is called a *semicircle*, remember that half of a sphere is generally called a *hemisphere*.

VOLUME OF A SPHERE

Volume of a Sphere

Characterization 3

The third description of the sphere enables us to find its volume. To accomplish this, we treat the sphere as the theoretical limit of an inscribed regular polyhedron whose number of faces n increases without limit. The polyhedron can be separated into n congruent pyramids; the center of the sphere is the vertex of each pyramid.

As n increases, the length of the altitude of each pyramid approaches the length of the radius of the sphere. Next we find the sum of the volumes of these pyramids, the limit of which is the volume of the sphere.

Volume of a Sphere

In Figure 9.43, one of the pyramids described in the preceding paragraph is shown.

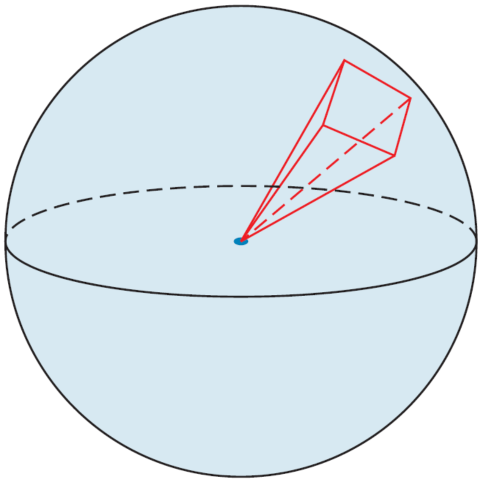


Figure 9.43

We designate the height of each and every pyramid by h.

Where the areas of the bases of the pyramids are written B_1, B_2, B_3, and so on, the sum of the volumes of the n pyramids forming the polyhedron is

$$\frac{1}{3}B_1h + \frac{1}{3}B_2h + \frac{1}{3}B_3h + \cdots + \frac{1}{3}B_nh$$

Volume of a Sphere

Next we write the volume of the polyhedron in the form

$$\frac{1}{3}h(B_1 + B_2 + B_3 + \cdots + B_n)$$

As n increases, $h \to r$ and $B_1 + B_2 + B_3 + \cdots + B_n \to S$, the surface area of the sphere. That is,

$$\frac{1}{3}h(B_1 + B_2 + B_3 + \cdots + B_n) \to \frac{1}{3}rS.$$

Because the surface area of the sphere is $S = 4\pi r^2$, the sum approaches the following limit as the volume of the sphere:

$$V = \frac{1}{3}r \cdot 4\pi r^2 = \frac{4}{3}\pi r^3$$

Volume of a Sphere

The preceding discussion suggests the following theorem.

Theorem 9.4.3

The volume *V* of a sphere with a radius of length *r* is given by

$$V = \frac{4}{3}\pi r^3.$$

Example 4

Find the exact volume of a sphere whose length of radius is 1.5 in.

Solution:

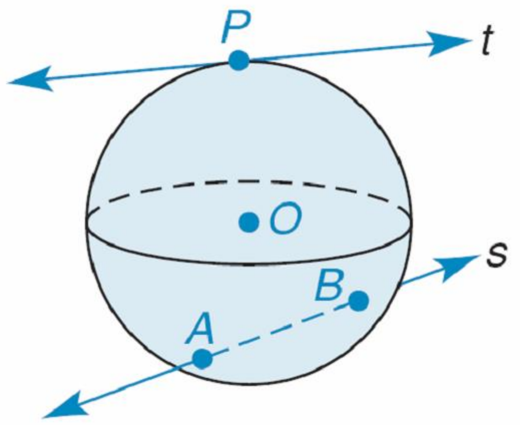
This calculation can be done more easily if we replace 1.5 by $\frac{3}{2}$.

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{\cancel{4}}{\cancel{3}} \cdot \pi \cdot \frac{\cancel{3}}{\cancel{2}} \cdot \frac{3}{\cancel{2}} \cdot \frac{3}{2} \\ &= \frac{9\pi}{2} \text{ in}^3 \end{aligned}$$

Volume of a Sphere

Just as two concentric circles have the same center but different lengths of radii, two spheres can also be concentric.

Like circles, spheres may have tangent lines and secant lines as illustrated in Figure 9.45(a).

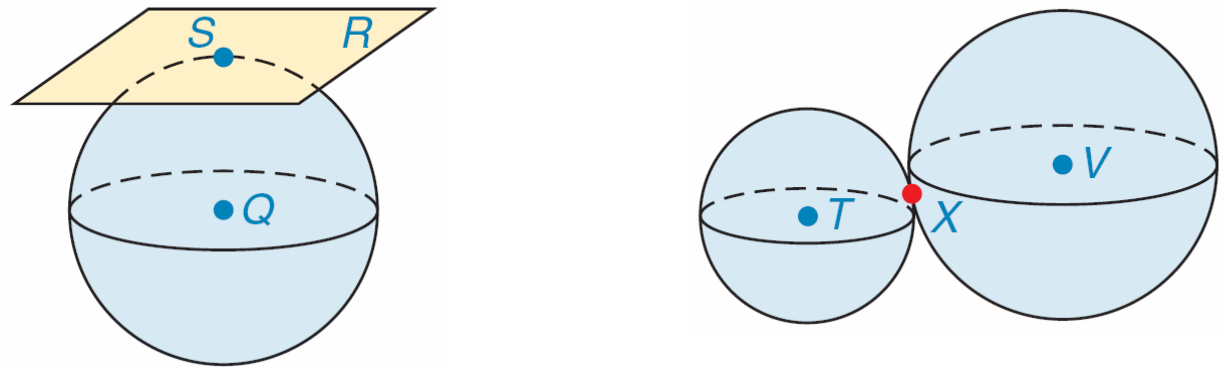


Line t is tangent to sphere O at point P; line s is a secant.

(a)

Figure 9.45

However, spheres also have tangent planes as shown in Figure 9.45(b). In Figure 9.45(c), spheres *T and V* are externally tangent; although no such drawing has been provided, two spheres may be internally tangent as well.



Plane *R* is tangent to sphere *Q* at point *S*.

(b)

Spheres *T* and *V* are externally tangent at point *X*.

(c)

Figure 9.45

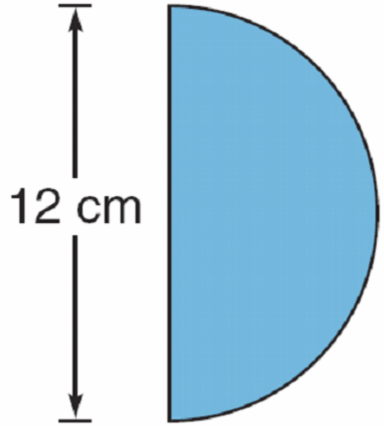
Volume of a Sphere

Each solid of revolution was generated by revolving a plane region about a horizontal line segment.

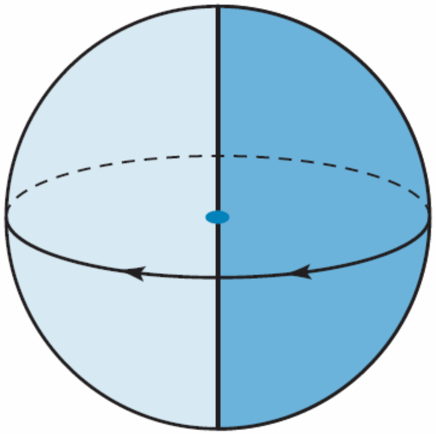
It is also possible to form a solid of revolution by rotating a region about a vertical or oblique line segment.

Example 7

Describe the solid of revolution that is formed when a semicircular region having a vertical diameter of length 12 cm [see Figure 9.46(a)] is revolved about that diameter. Then find the exact volume of the solid formed [see Figure 9.46(b)].



(a)



(b)

Figure 9.46

Example 7 – *Solution*

The solid that is formed is a sphere with length of radius $r = 6$ cm.

The formula we use to find the volume is $V = \frac{4}{3}\pi r^3$.
Then $V = \frac{4}{3}\pi \cdot 6^3$, which simplifies to $V = 288\pi$ cm^3.

More Solids of Revolution

When a circular region is revolved about a line in the circle's exterior, a doughnut shaped solid results.

The formal name of the resulting solid of revolution, shown in Figure 9.47, is the *torus*.

Methods of calculus are necessary to calculate both the surface area and the volume of the torus.

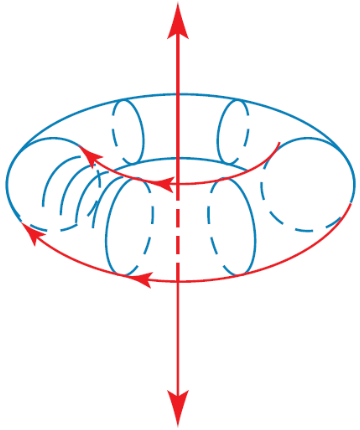


Figure 9.47