Chapter 9 Surfaces and Solids

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9.3 Cylinders and Cones

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CYLINDERS

Cylinders

Consider the solids in Figure 9.25, in which congruent circles lie in parallel planes. For the circles on the left, suppose that centers O and O' are joined to form $\overline{OO'}$; similarly, suppose that $\overline{QQ'}$ joins the centers of the circles on the right.

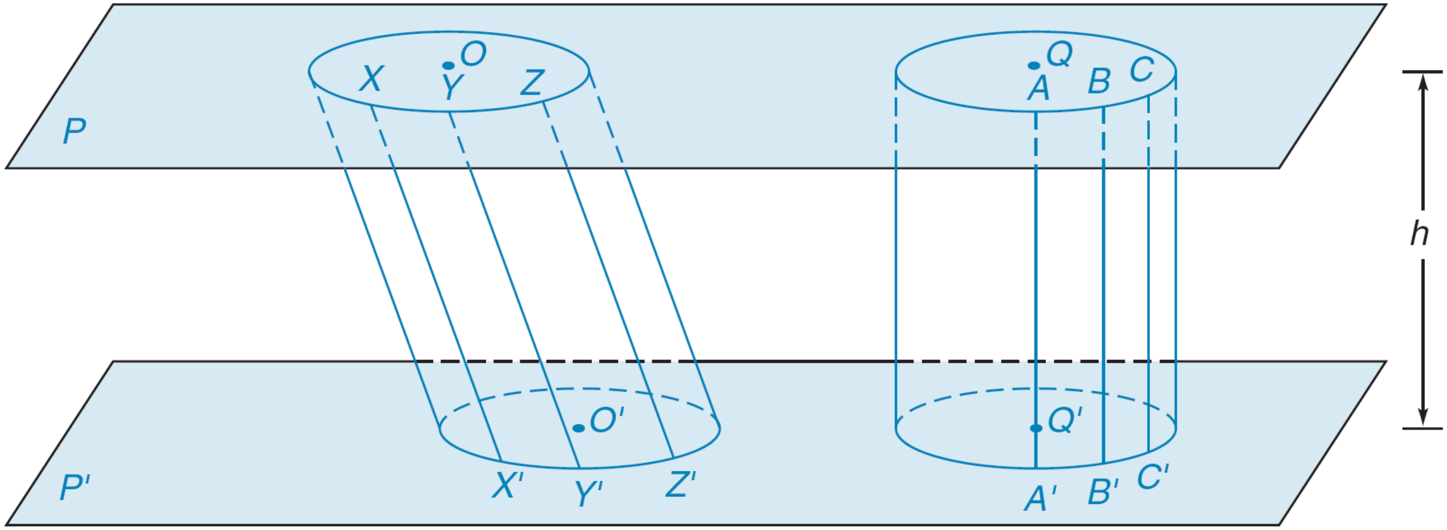


Figure 9.25

Cylinders

Let segments such as $\overline{XX'}$ join two points of the circles on the left, so that $\overline{XX'} \parallel \overline{OO'}$. If all such segments (like $\overline{XX'}$, $\overline{YY'}$, and $\overline{ZZ'}$) are parallel to each other, then a **cylinder** is generated.

Because $\overline{OO'}$ is not perpendicular to planes P and P', the solid on the left is an **oblique circular cylinder**.

With $\overline{QQ'}$ perpendicular to planes P and P', the solid on the right is a **right circular cylinder**.

Cylinders

For both cylinders, the distance *h* between the planes *P* and *P′* is the length of the **altitude** of the cylinder; *h* is also called the *height* of the cylinder.

The congruent circles are known as the **bases** of each cylinder.

Cylinders

A right circular cylinder is shown in Figure 9.26; however, the parallel planes are not pictured.

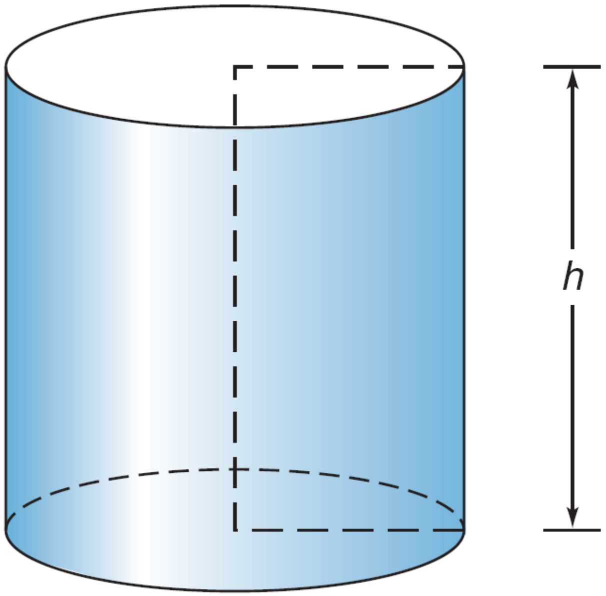


Figure 9.26

Cylinders

The line segment joining the centers of the two circular bases is known as the **axis** of the cylinder.

For a right circular cylinder, it is necessary that the axis be perpendicular to the planes of the circular bases; in such a case, the length of the altitude h is the length of the axis.

SURFACE AREA OF A CYLINDER

Surface Area of a Cylinder

The formula for the lateral area of a right circular cylinder (found in the following theorem) should be compared to the formula $L = hP$, the lateral area of a right prism whose base has perimeter P.

Theorem 9.3.1

The lateral area L of a right circular cylinder with altitude of length h and circumference C of the base is given by
$L = hC$.

Alternative Form: The lateral area of the right circular cylinder can be expressed in the form $L = 2\pi r h$, where r is the length of the radius of the circular base.

Surface Area of a Cylinder

Theorem 9.3.2

The total area T of a right circular cylinder with base area B and lateral area L is given by $T = L + 2B$.

Alternative Form: Where r is the length of the radius of the base and h is the length of the altitude of the cylinder, the total area can be expressed in the form $T = 2\pi rh + 2\pi r^2$.

Example 1

For the right circular cylinder shown in Figure 9.27, find the

a) exact lateral area L.

b) exact surface area T.

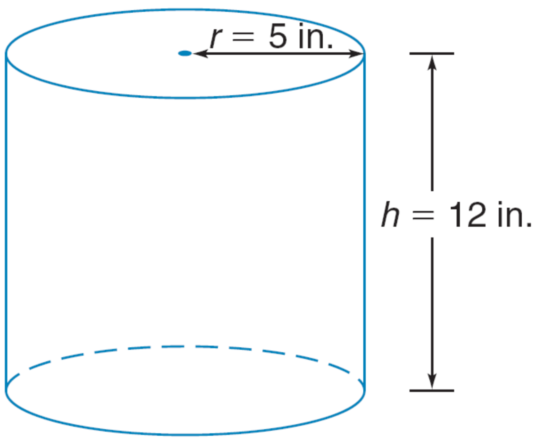


Figure 9.27

Solution:

a) $L = 2\pi r h$

$= 2 \cdot \pi \cdot 5 \cdot 12$

$= 120\pi \text{ in}^2$

Example 1 – *Solution*

cont'd

b) $T = L + 2B$

$= 2\pi r h + 2\pi r^2$

$= 120 \cdot \pi + 2 \cdot \pi \cdot 5^2$

$= 120\pi + 50\pi$

$= 170\pi \text{ in}^2$

VOLUME OF A CYLINDER

Volume of a Cylinder

In considering the volume of a right circular cylinder, recall that the volume of a prism is given by $V = Bh$, where B is the area of the base.

In Figure 9.28, we inscribe a prism in the cylinder as shown.

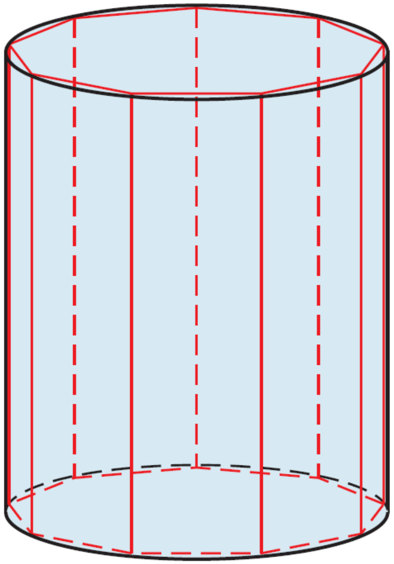


Figure 9.28

Volume of a Cylinder

Suppose that the prism is regular and that the number of sides in the inscribed polygon’s base becomes larger and larger; thus, the base approaches a circle in this limiting process.

The area of the polygonal base also approaches the area of the circle, and the volume of the prism approaches that of the right circular cylinder.

Volume of a Cylinder

Our conclusion is stated without proof in the following theorem.

Theorem 9.3.3
The volume V of a right circular cylinder with base area B and altitude of length h is given by $V = Bh$.

Alternative Form: Where r is the length of the radius of the base, the volume for the right circular cylinder can be written $V = \pi r^2 h$.

Example 2

If d = 4 cm and h = 3.5 cm, use a calculator to find the approximate volume of the right circular cylinder shown in Figure 9.29. Give the answer correct to two decimal places.

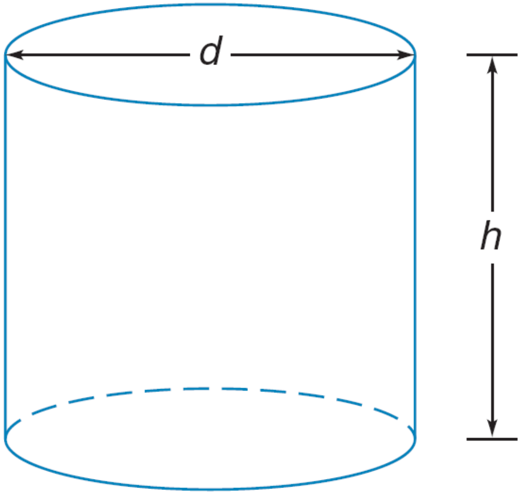


Figure 9.29

Example 2 – *Solution*

$$d = 4,$$

so $\quad r = 2.$

Thus, $V = Bh$ or $V = \pi r^2 h$ becomes

$$V = \pi \cdot 2^2(3.5)$$

$$= \pi \cdot 4(3.5)$$

$$= 14\pi$$

$$\approx 43.98 \text{ cm}^3$$

Volume of a Cylinder

Table 9.2 should help us recall and compare the area and volume formulas.

TABLE 9.2

	Lateral Area	Total Area	Volume
Prism	$L = hP$	$T = L + 2B$	$V = Bh$
Cylinder	$L = hC$	$T = L + 2B$	$V = Bh$

CONES

Cones

In Figure 9.30, consider point *P*, which lies outside the plane containing circle *O*.

A surface known as a **cone** results when line segments are drawn from *P* to points on the circle.

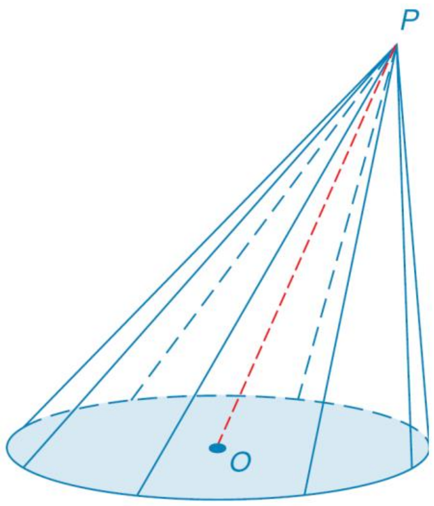


Figure 9.30

Cones

However, if P is joined to all possible points on the circle as well as to points in the interior of the circle, a solid is formed.

If $\overline{PO}$ is not perpendicular to the plane of circle O in Figure 9.30, the cone is an **oblique circular cone**.

In Figures 9.30 and 9.31, point P is the **vertex** (or **apex)** of the cone, and circle O is the **base**. The segment $\overline{PO}$, which joins the vertex to the cente circular base, is the **axis** of the cone.

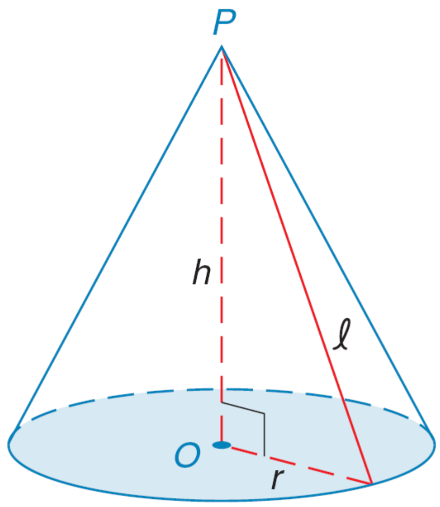


Figure 9.31

Cones

If the axis is perpendicular to the plane containing the base, as in Figure 9.31, the cone is a **right circular cone**.

In any cone, the perpendicular segment from the vertex to the plane of the base is the **altitude** of the cone.

In a right circular cone, the length h of the altitude equals the length of the axis.

For a right circular cone, and only for this type of cone, any line segment that joins the vertex to a point on the circle is a **slant height** of the cone; we will denote the length of the slant height by ℓ as shown in Figure 9.31.

SURFACE AREA OF A CONE

Surface Area of a Cone

We know that the lateral area for a regular pyramid is given by $L = \frac{1}{2}\ell P$.

For a right circular cone, consider an inscribed regular pyramid as in Figure 9.32.

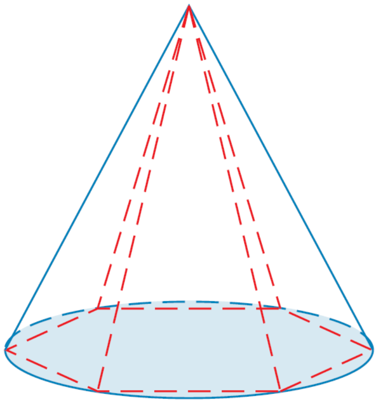


Figure 9.32

As the number of sides of the inscribed polygon's base grows larger, the perimeter of the inscribed polygon approaches the circumference of the circle as a limit.

Surface Area of a Cone

In addition, the slant height of the congruent triangular faces approaches that of the slant height of the cone.

Thus, the lateral area of the right circular cone can be compared to $L = \frac{1}{2}\ell P$; for the cone, we have

$$L = \frac{1}{2}\ell C$$

in which C is the circumference of the base. The fact that $C = 2\pi r$ leads to

$$L = \frac{1}{2}\ell(2\pi r)$$

so $$L = \pi r \ell$$

Surface Area of a Cone

Theorem 9.3.4

The lateral area L of a right circular cone with slant height of length ℓ and circumference C of the base is given by $L = \frac{1}{2}\ell C$.

Alternative Form: Where r is the length of the radius of the base, $L = \pi r \ell$.

Surface Area of a Cone

The following theorem follows easily from Theorem 9.3.4.

Theorem 9.3.5

The total area T of a right circular cone with base area B and lateral area L is given by $T = B + L$.

Alternative Form: Where r is the length of the radius of the base and ℓ is the length of the slant height, $T = \pi r^2 + \pi r \ell$.

Example 4

For the right circular cone in which $r = 3$ cm and $h = 6$ cm (see Figure 9.33), find the

a) exact and approximate lateral area L.
b) exact and approximate total area T.

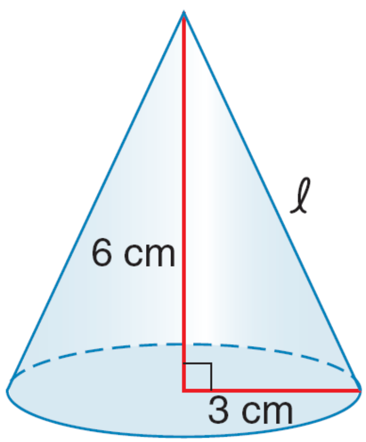


Figure 9.33

Example 4(a) – *Solution*

We need the length of the slant height ℓ for each problem part, so we apply the Pythagorean Theorem:

$$\ell^2 = r^2 + h^2$$

$$= 3^2 + 6^2$$

$$= 9 + 36$$

$$= 45$$

Then $\ell = \sqrt{45}$

$$= \sqrt{9 \cdot 5}$$

$$= \sqrt{9} \cdot \sqrt{5} = 3\sqrt{5}$$

Example 4(a) – *Solution*

cont'd

Using $L = \pi r \ell$, we have

$$L = \pi \cdot 3 \cdot 3\sqrt{5}$$

$$= 9\pi\sqrt{5}\ \text{cm}^2$$

$$\approx 63.22\ \text{cm}^2$$

Example 4(b) – *Solution*

cont'd

We also have

$$T = B + L$$

$$= \pi r^2 + \pi r \ell$$

$$= \pi \cdot 3^2 + \pi \cdot 3 \cdot 3\sqrt{5}$$

$$= (9\pi + 9\pi\sqrt{5}) \text{ cm}^2$$

$$\approx 91.50 \text{ cm}^2$$

Surface Area of a Cone

The following theorem was illustrated in the solution of Example 4.

Theorem 9.3.6

In a right circular cone, the lengths of the radius r (of the base), the altitude h, and the slant height ℓ satisfy the Pythagorean Theorem; that is, $\ell^2 = r^2 + h^2$ in every right circular cone.

VOLUME OF A CONE

Volume of a Cone

We know that the volume of a pyramid is given by the formula $V = \frac{1}{3}Bh$.

Consider a regular pyramid inscribed in a right circular cone. If its number of sides increases indefinitely, the volume of the pyramid approaches that of the right circular cone (see Figure 9.34).

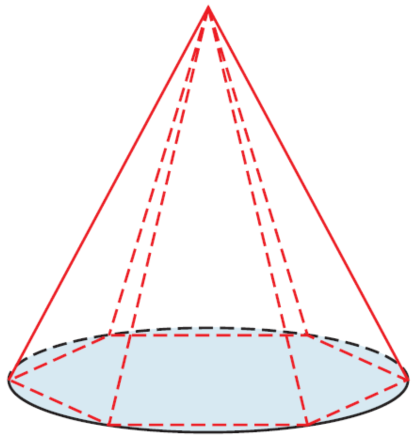


Figure 9.34

Volume of a Cone

Then the volume of the right circular cone is $V = \frac{1}{3}Bh$.

Because the area of the base of the cone is $B = \pi r^2$, an alternative formula for the volume of the cone is

$$V = \frac{1}{3}\pi r^2 h$$

We state this result as a theorem.

Volume of a Cone

Theorem 9.3.7

The volume V of a right circular cone with base area B and altitude of length h is given by

$$V = \frac{1}{3}Bh.$$

Alternative Form: Where r is the length of the radius of the base, the formula for the volume of the cone is usually written $V = \frac{1}{3}\pi r^2 h$.

Volume of a Cone

Table 9.3 should help us to recall and compare the area and volume formulas.

TABLE 9.3

	Lateral Area	Total Area	Volume	Slant Height
Pyramid	$L = \frac{1}{2}\ell P$	$T = B + L$	$V = \frac{1}{3}Bh$	$\ell^2 = a^2 + h^2$
Cone	$L = \frac{1}{2}\ell C$	$T = B + L$	$V = \frac{1}{3}Bh$	$\ell^2 = r^2 + h^2$

NOTE: The formulas that contain the slant height ℓ are used only with the regular pyramid and the right circular cone.

SOLIDS OF REVOLUTION

Solids of Revolution

Suppose that part of the boundary for a plane region is a line segment.

When the plane region is revolved about this line segment, the locus of points generated in space is called a **solid of revolution**.

The complete 360° rotation moves the region about the edge until the region returns to its original position.

The side (edge) used is called the **axis** of the resulting solid of revolution.

Example 5

Describe the solid of revolution that results when

a) a rectangular region with dimensions 2 ft by 5 ft is revolved about the 5-ft side [see Figure 9.35(a)].

b) a semicircular region with radius of length 3 cm is revolved about the diameter shown in Figure 9.35(b).

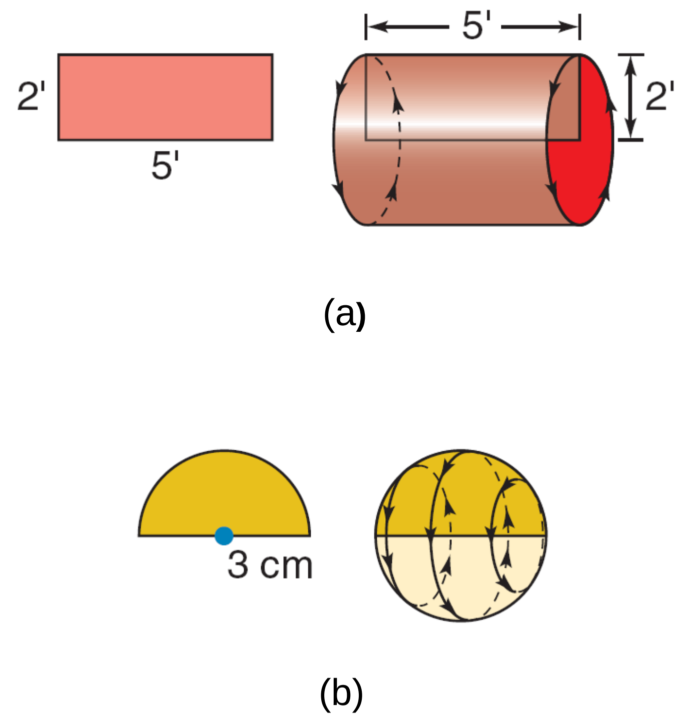


Figure 9.35

Example 5 – *Solution*

a) In Figure 9.35(a), the rectangle on the left is revolved about the 5-ft side to form the solid on the right.

The solid of revolution generated is a right circular cylinder that has a base radius of 2 ft and an altitude of 5 ft.

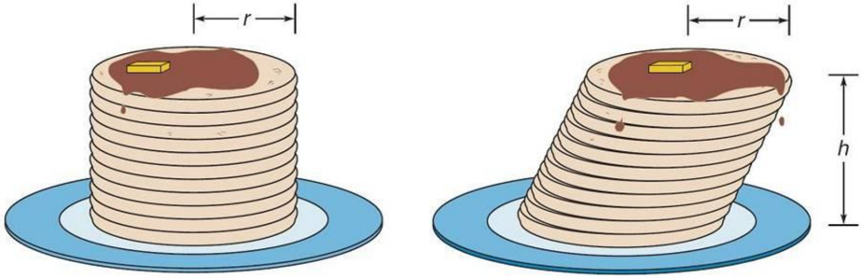
b) In Figure 9.35(b), the semicircle on the left is revolved about its diameter to form the solid on the right.

The solid of revolution generated is a *sphere* with a radius of length 3 cm.

Solids of Revolution

It may come as a surprise that the formulas that are used to calculate the volumes of an oblique circular cylinder and a right circular cylinder are identical.

To see why the formula $V = Bh$ or $V = \pi r^2 h$ can be used to calculate the volume of an oblique circular cylinder, consider the stacks of pancakes shown in Figures 9.37(a) and 9.37(b).



(a) (b)

Figure 9.37

Solids of Revolution

With each stack *h* units high, the volume is the same regardless of whether the stack is vertical or oblique.

It is also true that the formula for the volume of an oblique circular cone is the same as the formula for the volume of the right circular cone.

For both the right circular cone and the oblique circular cone, $V = \frac{1}{3}Bh$ or $V = \frac{1}{3}\pi r^2 h$.