Chapter 9 Surfaces and Solids

9.2 Pyramids, Area, and Volume

Pyramids, Area, and Volume

The solids (space figures) shown in Figure 9.14 below are **pyramids**.

In Figure 9.14(a), point *A* is noncoplanar with square *base BCDE*. In Figure 9.14(b), *F* is noncoplanar with its base, $\triangle GHJ$.

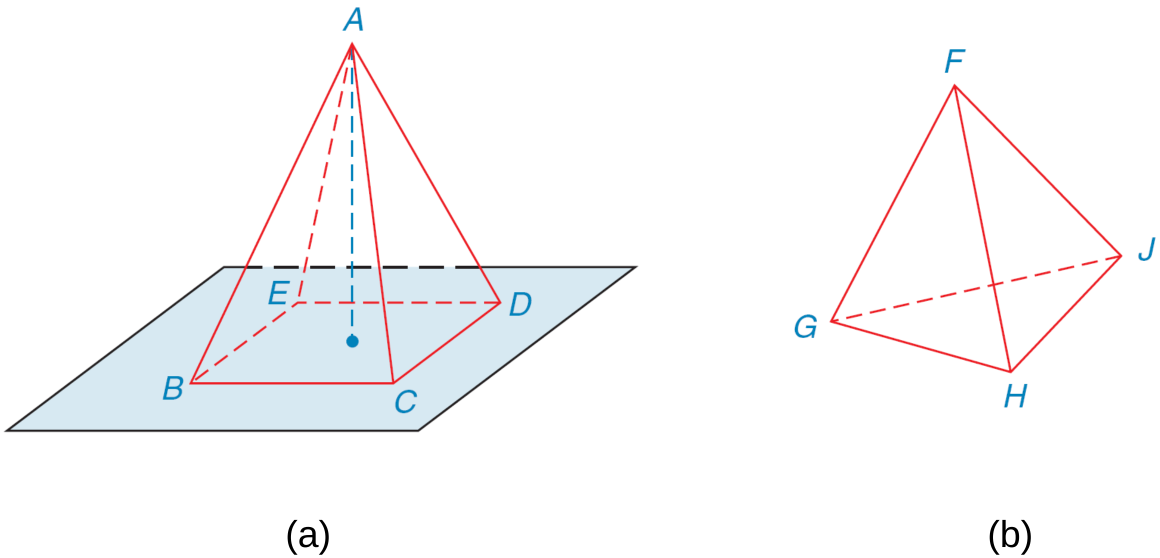


Figure 9.14

Pyramids, Area, and Volume

In each space pyramid, the noncoplanar point is joined to each vertex as well as each point of the base.

A solid pyramid results when the noncoplanar point is joined both to points on the polygon as well as to points in its interior.

Point *A* is known as the **vertex** or **apex** of the **square pyramid**; likewise, point *F* is the vertex or apex of the **triangular pyramid**.

The pyramid of Figure 9.14(b) has *four* triangular faces; for this reason, it is called a **tetrahedron**.

Pyramids, Area, and Volume

The pyramid in Figure 9.15 is a **pentagonal pyramid**.

It has vertex *K*, pentagon *LMNPQ* for its **base**, and **lateral edges** $\overline{KL}$, $\overline{KM}$, $\overline{KN}$, $\overline{KP}$, and $\overline{KQ}$.

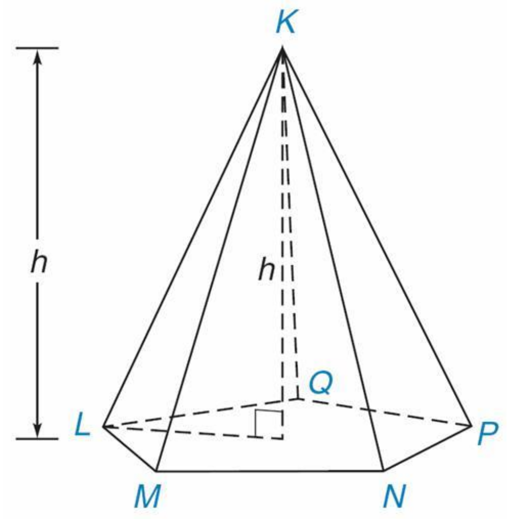


Figure 9.15

Although *K* is called *the* **vertex of the pyramid**, there are actually six vertices: *K*, *L*, *M*, *N*, *P*, and *Q*.

The sides of the base $\overline{LM}$, $\overline{MN}$, $\overline{NP}$, $\overline{PQ}$, and $\overline{QL}$ are **base edges**.

Pyramids, Area, and Volume

All **lateral faces** of a pyramid are triangles; △ *KLM* is one of the five lateral faces of the pentagonal pyramid.

Including base *LMNPQ*, this pyramid has a total of six faces. The **altitude** of the pyramid, of length *h*, is the line segment from the vertex *K* perpendicular to the plane of the base.

Definition

A **regular pyramid** is a pyramid whose base is a regular polygon and whose lateral edges are all congruent.

Pyramids, Area, and Volume

Suppose that the pyramid in Figure 9.15 is a regular pentagonal pyramid.

Then the lateral faces are necessarily congruent to each other; by SSS, $\triangle KLM \cong \triangle KMN \cong \triangle KNP \cong \triangle KPQ \cong \triangle KQL$.

Each lateral face is an isosceles triangle. In a regular pyramid, the altitude joins the apex of the pyramid to the center of the regular polygon that is the base of the pyramid.

The length of the altitude is height *h*.

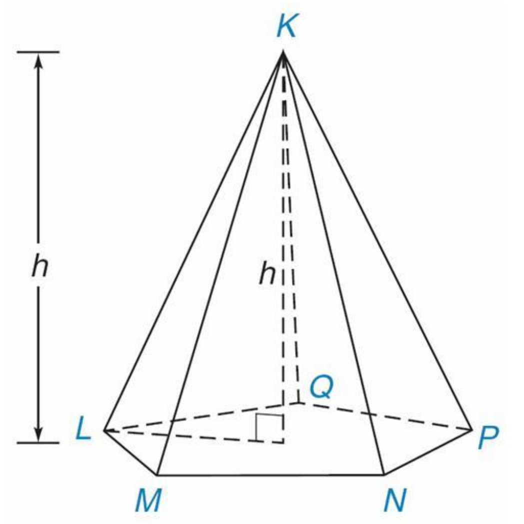


Figure 9.15

Pyramids, Area, and Volume

Definition

The **slant height** of a regular pyramid is the altitude from the vertex (apex) of the pyramid to the base of any of the congruent lateral faces of the regular pyramid.

In our formulas and explanations, we use ℓ to represent the length of the slant height of a regular pyramid. See Figure 9.16(c).

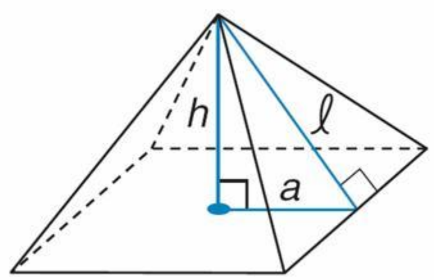


Figure 9.16(c)

Example 1

For a regular square pyramid with height 4 in. and base edges of length 6 in. each, find the length of the slant height ℓ. (See Figure 9.16.)

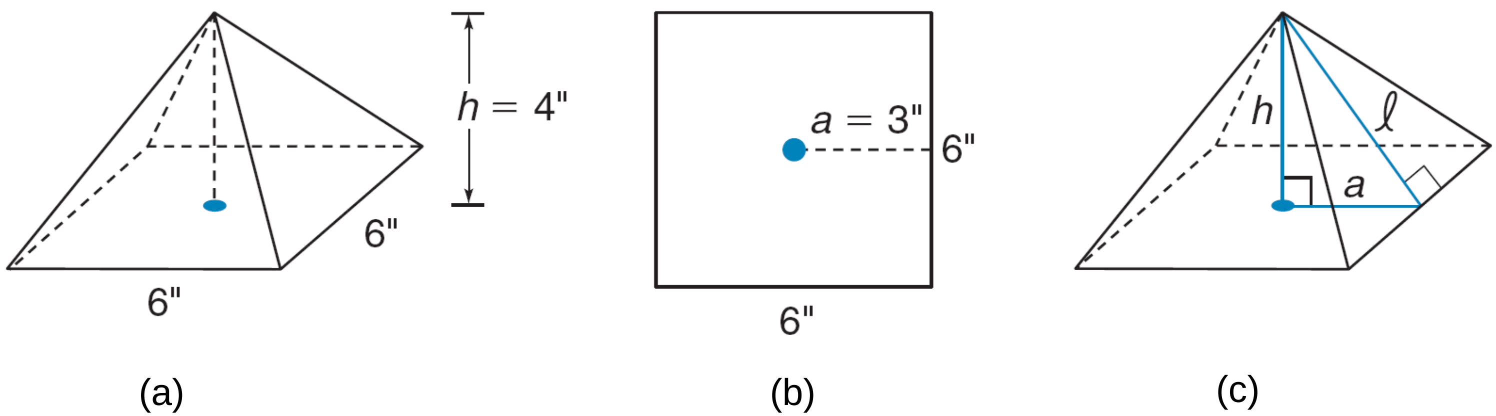


Figure 9.16

Example 1 – *Solution*

In Figure 9.16, it can be shown that the apothem to any side has length 3 in. (one-half the length of the side of the square base).

Also, the slant height is the hypotenuse of a right triangle with legs equal to the lengths of the altitude of the pyramid and the apothem of the base.

See Figure 9.16(c). Applying the Pythagorean Theorem,

$$\ell^2 = a^2 + h^2$$

$$\ell^2 = 3^2 + 4^2$$

Example 1 – *Solution*

cont'd

$$\ell^2 = 9 + 16$$

$$\ell^2 = 25$$

$$\ell = 5 \text{ in.}$$

Pyramids, Area, and Volume

Theorem 9.2.1

In a regular pyramid, the lengths of the apothem a of the base, the altitude h, and the slant height ℓ satisfy the Pythagorean Theorem; that is, $\ell^2 = a^2 + h^2$.

SURFACE AREA OF A PYRAMID

Surface Area of a Pyramid

To lay the groundwork for the next theorem, we justify the result by “taking apart” one of the regular pyramids and laying it out flat.

Although we use a regular hexagonal pyramid for this purpose, the argument is similar if the base is any regular polygon.

Surface Area of a Pyramid

When the lateral faces of the regular pyramid are folded down into the plane, as shown in Figure 9.17, the shaded lateral area is the sum of the areas of the congruent triangular lateral faces.

Using $A = \frac{1}{2}bh$, we find that the area of each triangular face is $\frac{1}{2} \cdot s \cdot \ell$ (each side of the base of the pyramid has length s, and the slant height has length ℓ).

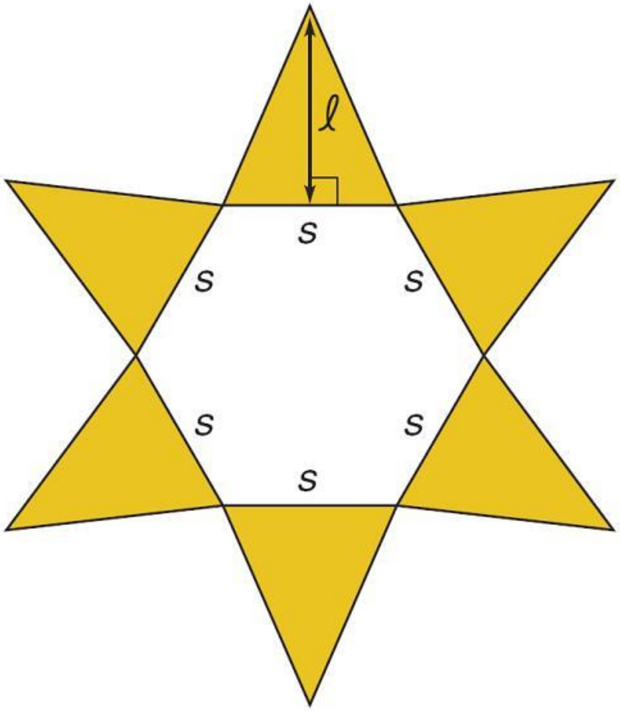


Figure 9.17

Surface Area of a Pyramid

The combined areas of the triangles give the lateral area. Because there are *n* triangles,

$$L = n \cdot \frac{1}{2} \cdot s \cdot \ell$$

$$= \frac{1}{2} \cdot \ell(n \cdot s)$$

$$= \frac{1}{2}\ell P$$

where *P* is the perimeter of the base.

Surface Area of a Pyramid

Theorem 9.2.2

The lateral area L of a regular pyramid with slant height of length ℓ and perimeter P of the base is given by

$$L = \frac{1}{2}\ell P$$

Example 2

Find the lateral area of the regular pentagonal pyramid in Figure 9.18(a) if the sides of the base measure 8 cm and the lateral edges measure 10 cm each.

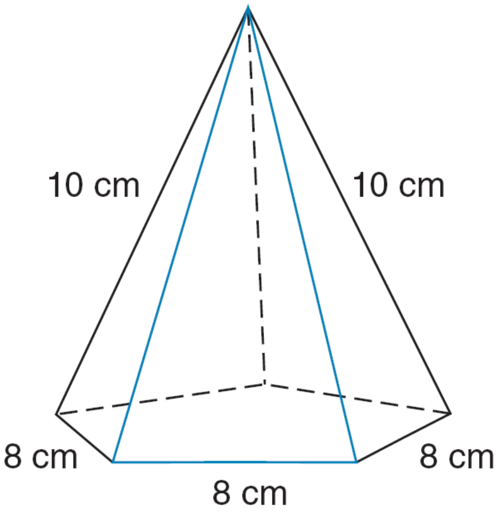


Figure 9.18(a)

Example 2 – *Solution*

For the triangular lateral face in Figure 9.18(b), the slant height bisects the base edge as indicated.

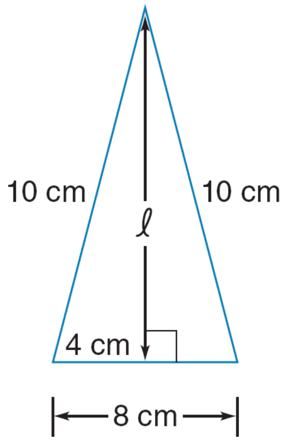


Figure 9.18(b)

Applying the Pythagorean Theorem, we have

$$4^2 + \ell^2 = 10^2,$$

Example 2 – *Solution*

cont'd

so $16 + \ell^2 = 100$

$$\ell^2 = 84$$

$$\begin{aligned} \ell &= \sqrt{84} \\ &= \sqrt{4 \cdot 21} \\ &= \sqrt{4} \cdot \sqrt{21} \\ &= 2\sqrt{21} \end{aligned}$$

Now $L = \frac{1}{2}\ell P$ becomes $L = \frac{1}{2} \cdot 2\sqrt{21} \cdot (5 \cdot 8)$

Example 2 – *Solution*

cont'd

$$= \frac{1}{2} \cdot 2\sqrt{21} \cdot 40$$

$$= 40\sqrt{21} \text{ cm}^2$$

$\approx$ 183.30 cm^2.

Surface Area of a Pyramid

It may be easier to find the lateral area of a regular pyramid without using the formula of Theorem 9.2.2; simply find the area of one lateral face and multiply by the number of faces.

In Example 2, the area of each triangular face is $\frac{1}{2} \cdot 8 \cdot 2\sqrt{21}$ or $8\sqrt{21}$; thus, the lateral area of the regular pentagonal pyramid is $5 \cdot 8\sqrt{21} = 40\sqrt{21}$ cm^2.

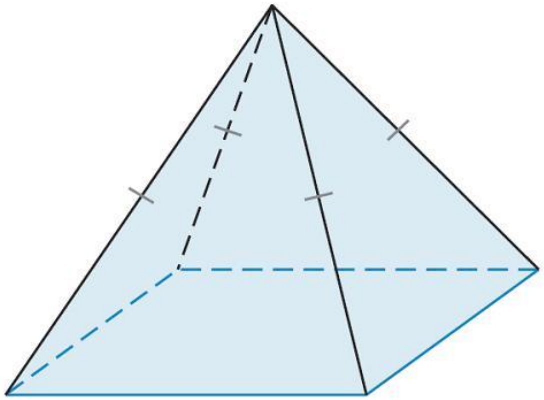
Surface Area of a Pyramid

Theorem 9.2.3

The total area (surface area) T of a pyramid with lateral area L and base area B is given by $T = L + B$.

The formula for the total area T of the pyramid can be written $T = \frac{1}{2}\ell P + B$.

The pyramid in Figure 9.20(a) is a regular square pyramid rather than just a square pyramid. It has congruent lateral edges and congruent lateral faces.



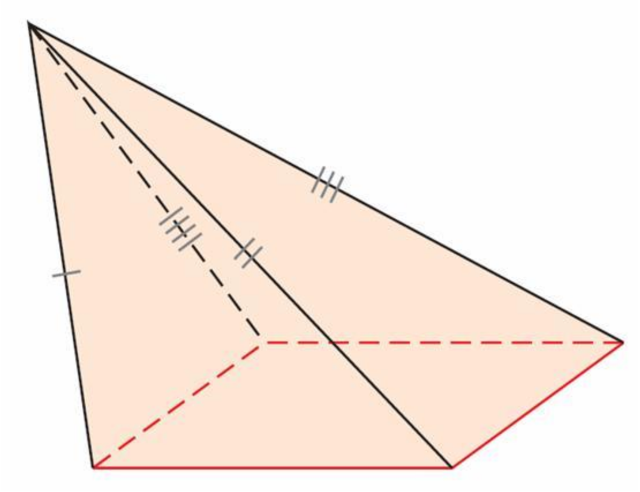
Regular square pyramid

Figure 9.20(a)

Surface Area of a Pyramid

The pyramid shown in Figure 9.20(b) is oblique.

The oblique square pyramid has neither congruent lateral edges nor congruent lateral faces.



Square pyramid

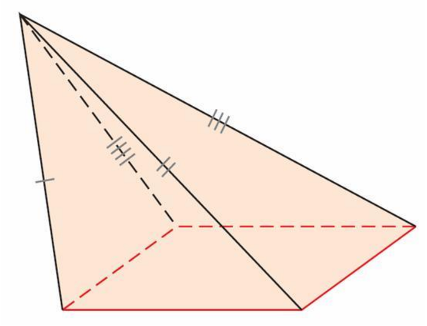
Figure 9.20(b)

VOLUME OF A PYRAMID

Volume of a Pyramid

The factor "one third" in the formula for the volume of a pyramid provides exact results.

This formula can be applied to any pyramid, even one that is not regular; in Figure 9.20(b), the length of the altitude is the perpendicular distance from the vertex to the plane of the square base.



Square pyramid

Figure 9.20(b)

Volume of a Pyramid

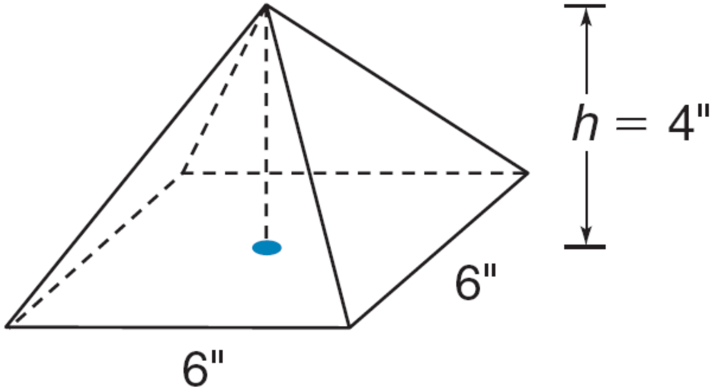
Theorem 9.2.4

The volume V of a pyramid having a base area B and an altitude of length h is given by

$$V = \frac{1}{3}Bh$$

Example 4

Find the volume of the regular square pyramid with height h = 4 in. and base edges of length s = 6 in.



Example 4 – *Solution*

The area of the square base is $B = (6 \text{ in.})^2$ or 36 in^2.

Because $h = 4$ in., the formula $V = \frac{1}{3} Bh$ becomes

$$V = \frac{1}{3}(36 \text{ in}^2)(4 \text{ in.})$$

$$= 48 \text{ in}^3$$

Volume of a Pyramid

To find the volume of a pyramid by using the formula $V = \frac{1}{3}Bh$, it is often necessary to determine *B* or *h* from other information that has been provided.

Table 9.1 reminds us of the types of units necessary in different types of measure.

TABLE 9.1

Type of Measure	Geometric Measure	Type of Unit
Linear	Length of segment, such as length of slant height	in., cm, etc.
Area	Amount of plane region enclosed, such as area of lateral face	in^2, cm^2, etc.
Volume	Amount of space enclosed, such as volume of a pyramid	in^3, cm^3, etc.

Volume of a Pyramid

Theorem 9.2.5

In a regular pyramid, the lengths of altitude h, radius r of the base, and lateral edge e satisfy the Pythagorean Theorem; that is, $e^2 = h^2 + r^2$.

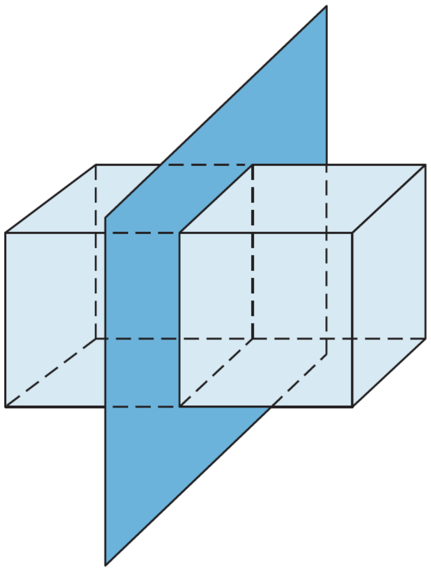
Plane and solid figures may have line symmetry and point symmetry. However, solid figures may also have **plane symmetry**.

To have this type of symmetry, a plane can be drawn for which each point of the space figure has a corresponding point on the opposite side of the plane at the same distance from the plane.

Volume of a Pyramid

Each solid in Figure 9.24 has more than one plane of symmetry.

In Figure 9.24(a), the plane of symmetry shown is determined by the midpoints of the indicated edges of the "box."

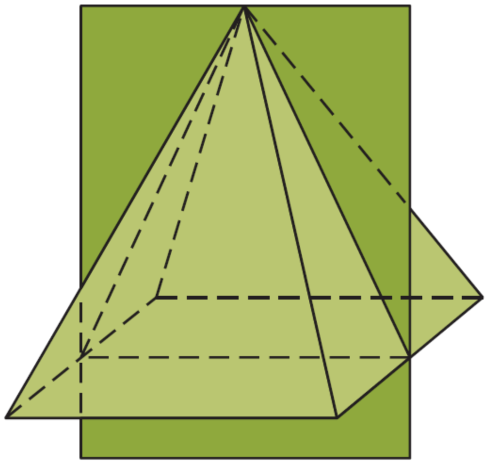


Right rectangular prism

Figure 9.24(a)

Volume of a Pyramid

In Figure 9.24(b), the plane determined by the apex and the midpoints of opposite sides of the square base leads to plane symmetry for the pyramid.



Regular square pyramid

Figure 9.24(b)