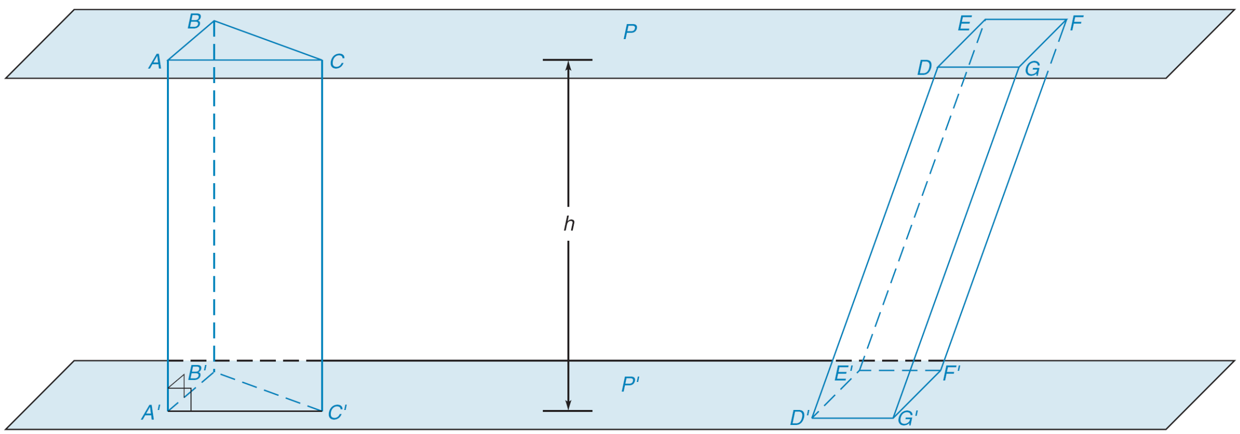
Chapter 9 Surfaces and Solids

9.1 Prisms, Area, and Volume

PRISMS

Prisms

Suppose that two congruent polygons lie in parallel planes in such a way that their corresponding sides are parallel. If the corresponding vertices of these polygons [such as *A* and *A′* in Figure 9.1(a)] are joined by line segments, then the "solid" or "space figure" that results is known as a **prism**.



$\Delta ABC \cong \Delta A'B'C'$

(a)

Square $DEFG \cong$ Square $D'E'F'G'$

(b)

Figure 9.1

Prisms

The congruent figures that lie in the parallel planes are the **bases** of the prism.

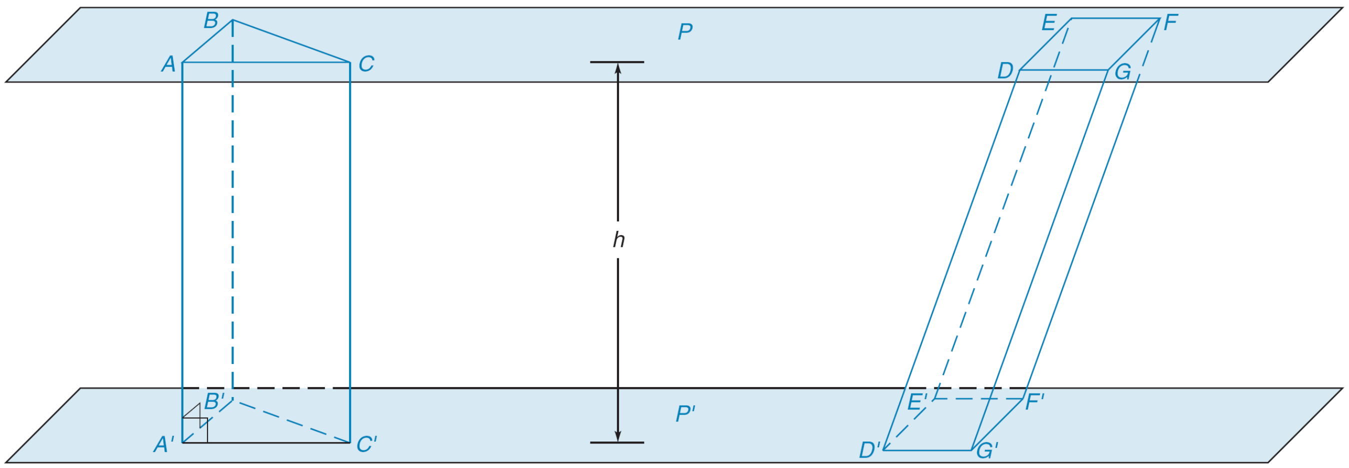
The parallel planes need not be shown in the drawings of prisms.

Suggested by an empty box, the prism is like a shell that encloses a portion of space by the parts of planes that form the prism; thus, a prism does not contain interior points.

In practice, it is sometimes convenient to call a prism such as a brick a *solid*; of course, this interpretation of prism would contain its interior points.

Prisms

In Figure 9.1(a), $\overline{AB}$, $\overline{AC}$, $\overline{BC}$, $\overline{A'B'}$, $\overline{A'C'}$, and $\overline{B'C'}$ are **base edges** and $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$ are **lateral edges** of the prism. Because the lateral edges of this prism are perpendicular to its base edges, the **lateral faces** (like quadrilateral *ACC′A′*) are rectangles.



$\Delta ABC \cong \Delta A'B'C'$
(a)

Square *DEFG* ≅ Square *D′E′F′G′*
(b)

Figure 9.1

Prisms

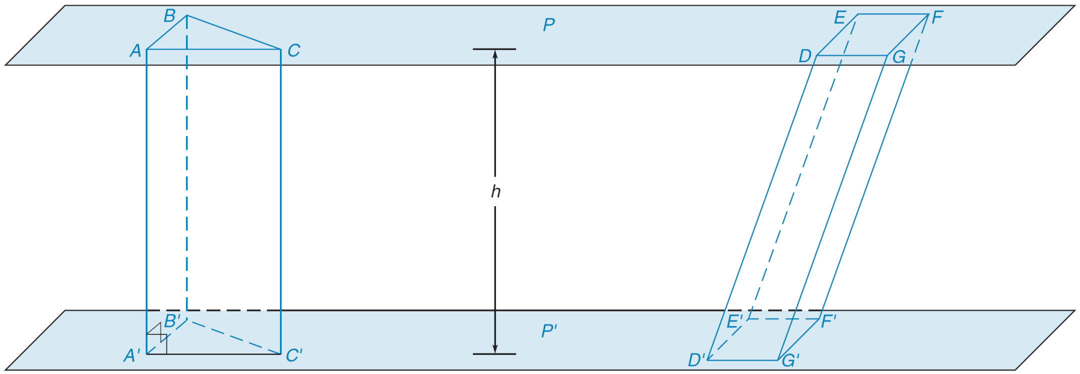
The bases and lateral faces are known collectively as the **faces** of the prism. Any point at which three faces are concurrent is a **vertex** of the prism. Points A, B, C, A', B', and C' are the **vertices** of the prism.

In Figure 9.1(b), the lateral edges of the prism are not perpendicular to its base edges; in this situation, the lateral edges are often described as **oblique** (slanted).

For the oblique prism, the lateral faces are parallelograms.

Prisms

Considering the prisms in Figure 9.1, we are led to the following definitions.



$\Delta ABC \cong \Delta A'B'C'$
(a)

Square $DEFG \cong$ Square $D'E'F'G'$
(b)

Figure 9.1

Definition

A **right prism** is a prism in which the lateral edges are perpendicular to the base edges at their points of intersection.

Prisms

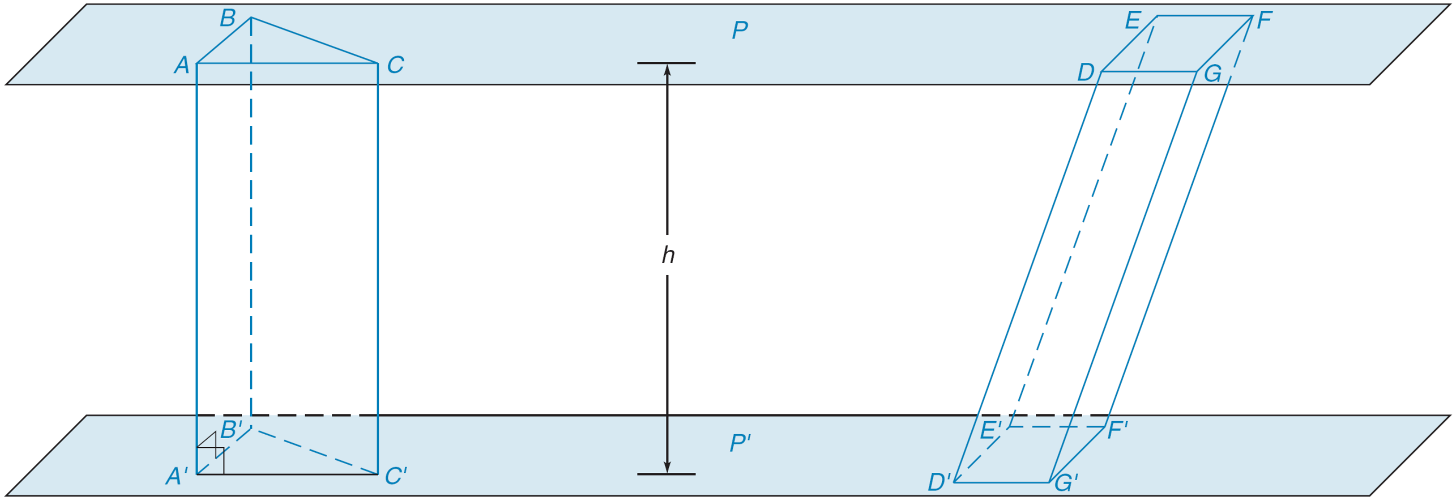
An **oblique prism** is a prism in which the parallel lateral edges are oblique to the base edges at their points of intersection.

Part of the description used to classify a prism depends on its base.

For instance, the prism in Figure 9.1(a) is a *right triangular prism*; in this case, the word *right* describes the prism, whereas the word *triangular* refers to the triangular base.

Prisms

Similarly, the prism in Figure 9.1(b) is an *oblique square prism*.



$\Delta ABC \cong \Delta A'B'C'$

(a)

Square $DEFG \cong$ Square $D'E'F'G'$

(b)

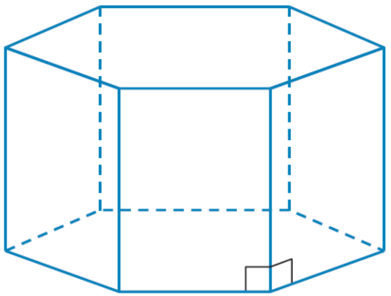
Figure 9.1

Prisms

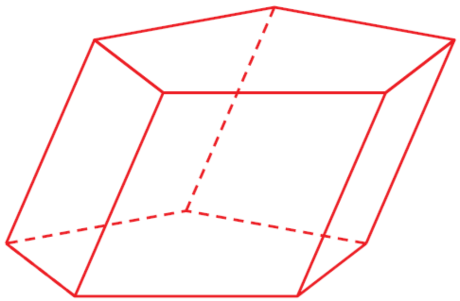
Both prisms in Figure 9.1 have an **altitude** (a perpendicular segment joining italics on height the planes that contain the bases) of length h, also known as the *height* of the prism.

Example 1

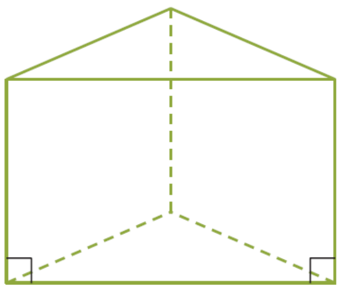
Name each type of prism in Figure 9.2.



(a)



(b)



Bases are equilateral triangles

(c)

Figure 9.2

Solution:

a) The lateral edges are perpendicular to the base edges of the hexagonal base. The prism is a *right hexagonal prism*.

Example 1 – *Solution*

cont'd

b) The lateral edges are oblique to the base edges of the pentagonal base. The prism is an *oblique pentagonal prism*.

c) The lateral edges are perpendicular to the base edges of the triangular base. Because the base is equilateral, the prism is a *right equilateral triangular prism*.

AREA OF A PRISM

Area of a Prism

Definition

The **lateral area *L*** of a prism is the sum of the areas of all lateral faces.

In the right triangular prism of Figure 9.3, *a*, *b*, and *c* are the lengths of the sides of either base.

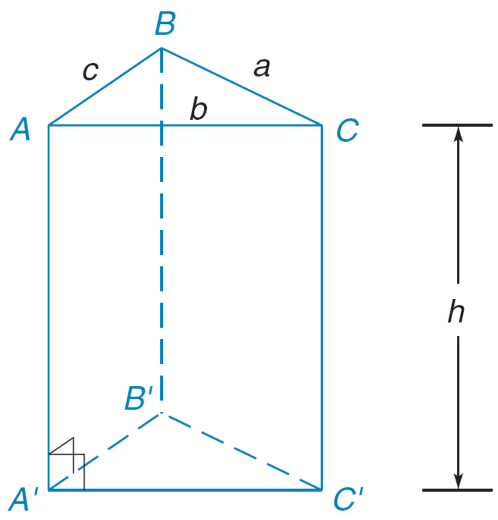


Figure 9.3

Area of a Prism

These dimensions are used along with the length of the altitude (denoted by h) to calculate the lateral area, the sum of the areas of rectangles $ACC'A'$, $ABB'A'$ and $BCC'B'$.

The lateral area L of the right triangular prism can be found as follows:

$$\begin{aligned} L &= ah + bh + ch \\ &= h(a + b + c) \\ &= hP \end{aligned}$$

where P is the perimeter of a base of the prism.

Area of a Prism

This formula, $L = hP$, is valid for finding the lateral area of any *right* prism.

Although lateral faces of an oblique prism are parallelograms, the formula $L = hP$ is also used to find its lateral area.

Theorem 9.1.1

The lateral area L of any prism whose altitude has measure h and whose base has perimeter P is given by $L = hP$.

Area of a Prism

Definition

For any prism, the **total area** ***T*** is the sum of the lateral area and the areas of the bases.

We know that the bases and lateral faces are known as *faces* of a prism.

Thus, the total area *T* of the prism is the sum of the areas of all its faces.

Area of a Prism

Theorem 9.1.2

The total area T of any prism with lateral area L and base area B is given by $T = L + 2B$.

Area of a Prism

Recalling Heron's Formula, we know that the base area *B* of the right triangular prism in Figure 9.6 can be found by the formula

$$B = \sqrt{s(s - a)(s - b)(s - c)}$$

in which *s* is the semiperimeter of the triangular base.

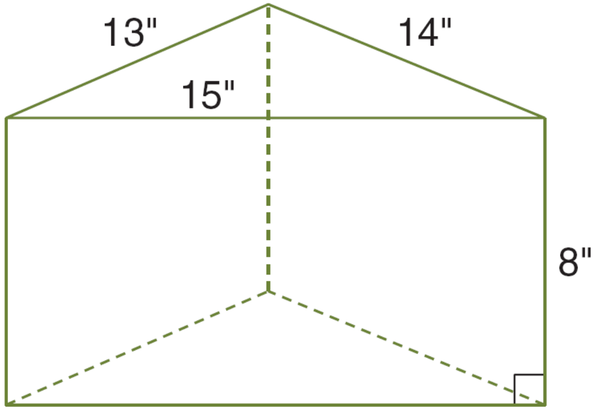


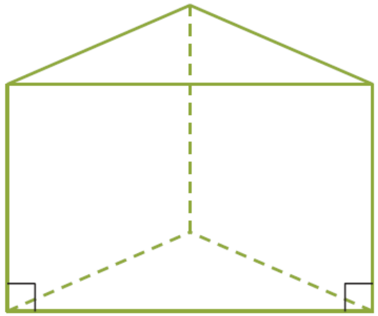
Figure 9.6

Area of a Prism

Definition

A **regular prism** is a right prism whose bases are regular polygons.

Consider this definition, the prism in Figure 9.2(c) will be called a regular triangular prism.



Bases are equilateral triangles

Figure 9.2(c)

Area of a Prism

Definition

A **cube** is a right square prism whose edges are congruent.

The cube is very important in determining the volume of a solid.

VOLUME OF A PRISM

Volume of a Prism

To introduce the notion of *volume*, we realize that a prism encloses a portion of space.

Without a formal definition, we say that **volume** is a number that measures the amount of enclosed space.

To begin, we need a unit for measuring volume.

Just as the meter can be used to measure length and the square yard can be used to measure area, a **cubic unit** is used to measure the amount of space enclosed within a bounded region of space.

Volume of a Prism

One such unit is described in the following paragraph.

The volume enclosed by the cube shown in Figure 9.9 is 1 cubic inch or 1 in^3. The volume of a solid is the number of cubic units within the solid.

Thus, we assume that the volume of any solid is a positive number of cubic units.

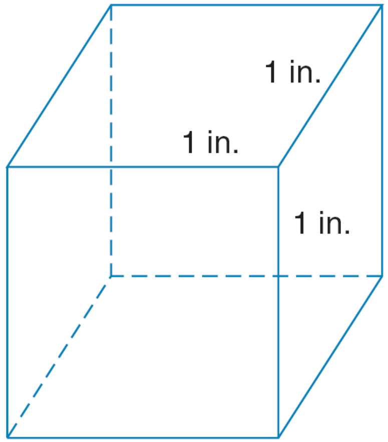


Figure 9.9

Volume of a Prism

Postulate 24 (Volume Postulate)
Corresponding to every solid is a unique positive number V known as the volume of that solid.

The simplest figure for which we can determine volume is the **right rectangular prism**.

Such a solid might be described as a **parallelpiped** or as a "box." Because boxes are used as containers for storage and shipping (such as a boxcar), it is important to calculate volume as a measure of capacity.

Volume of a Prism

A right rectangular prism is shown in Figure 9.10; its dimensions are length ℓ, width w, and height (or altitude) h.

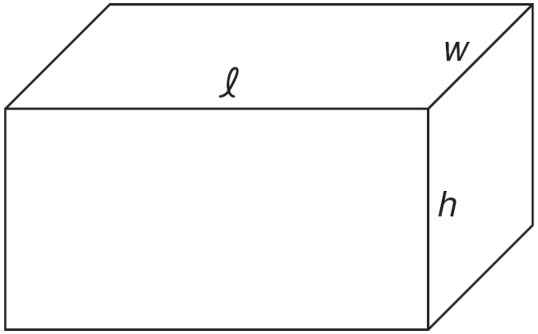


Figure 9.10

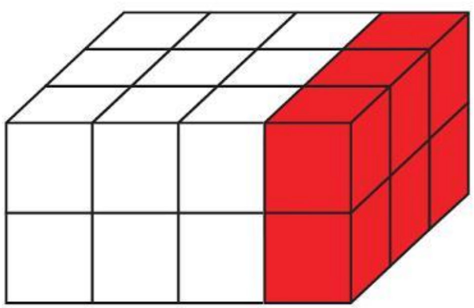
The volume of a right rectangular prism of length 4 in., width 3 in., and height 2 in. is easily shown to be 24 in^3.

The volume is the product of the three dimensions of the given solid.

Volume of a Prism

We see not only that $4 \cdot 3 \cdot 2 = 24$ but also that the units of volume are in. • in. • in. = in^3.

Figures 9.11(a) and (b) illustrate that the 4 by 3 by 2 box must have the volume 24 in^3.



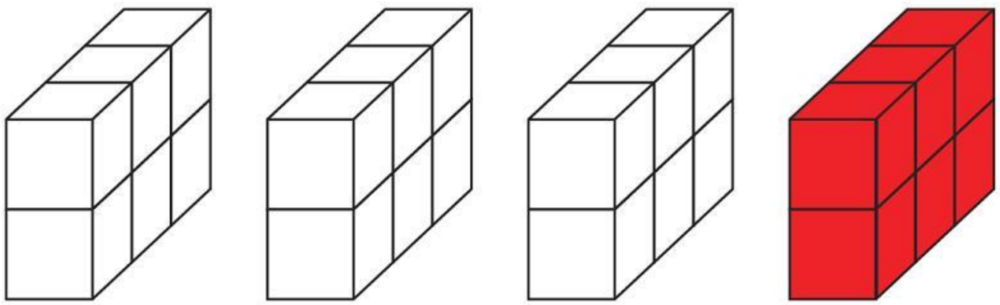


Figure 9.11

We see that there are four layers of blocks, each of which is a 2 by 3 configuration of 6 in^3.

Volume of a Prism

Figure 9.11 provides the insight that leads us to our next postulate.

Postulate 25

The volume of a right rectangular prism is given by

$$V = \ell w h$$

where ℓ measures the length, w the width, and h the altitude of the prism.

In order to apply the formula found in Postulate 25, the units used for dimensions ℓ, w, and h must be alike.

Example 6

Find the volume of a box whose dimensions are 1 ft, 8 in., and 10 in. (See Figure 9.12.)

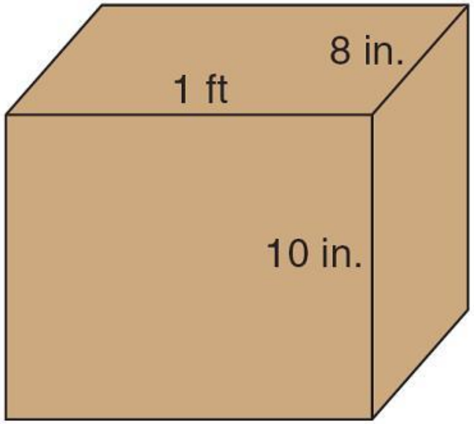


Figure 9.12

Solution:

Although it makes no difference which dimension is chosen for ℓ or w or h, it is most important that the units of measure be the same.

Example 6 – *Solution*

cont'd

Thus, 1 ft is replaced by 12 in. in the formula for volume:

$$V = \ell w h$$

$$= 12 \text{ in.} \cdot 8 \text{ in.} \cdot 10 \text{ in.}$$

$$= 960 \text{ in}^3$$

Volume of a Prism

The formula for the volume of the right rectangular prism, $V = \ell wh$, could be replaced by the formula $V = Bh$, where B is the area of the base of the prism; for a rectangular prism, $B = \ell w$.

As stated in the next postulate, this volume relationship is true for right prisms in general.

Volume of a Prism

Postulate 26

The volume of a right prism is given by

$$V = Bh$$

where B is the area of a base and h is the length of the altitude of the prism.

In real-world applications, the formula $V = Bh$ is valid for calculating the volumes of oblique prisms as well as right prisms.