Chapter 8

Areas of Polygons and Circles

8.2 Perimeter and Area of Polygons

Perimeter and Area of Polygons

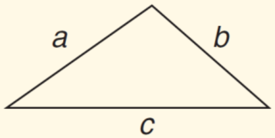
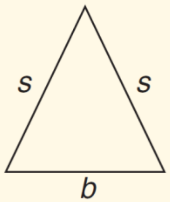
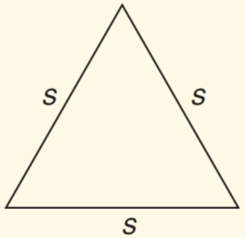
Definition

The **perimeter** of a polygon is the sum of the lengths of all sides of the polygon.

Table 8.1 summarizes perimeter formulas for types of triangles.

TABLE 8.1

Perimeter of a Triangle

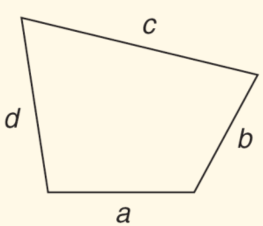
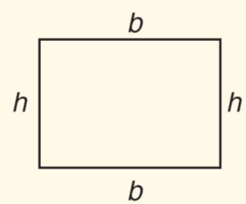
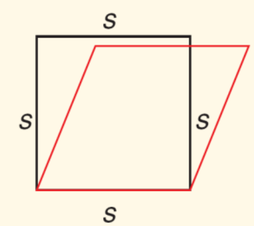
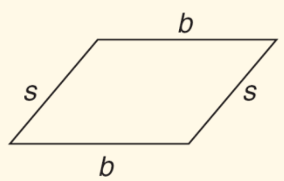
Scalene Triangle	Isosceles Triangle	Equilateral Triangle
 $P = a + b + c$	 $P = b + 2s$	 $P = 3s$

Perimeter and Area of Polygons

Table 8.2 summarizes formulas for the perimeters of selected types of quadrilaterals.

TABLE 8.2

Perimeter of a Quadrilateral

Quadrilateral	Rectangle	Square (or Rhombus)	Parallelogram
			
$P = a + b + c + d$	$P = 2b + 2h$ or $P = 2(b + h)$	$P = 4s$	$P = 2b + 2s$ or $P = 2(b + s)$

However, it is more important to understand the concept of perimeter than to memorize formulas.

Example 1

Find the perimeter of $\triangle ABC$ shown in Figure 8.17 if:

a) $AB = 5$ in., $AC = 6$ in., and $BC = 7$ in.

b) $AD = 8$ cm, $BC = 6$ cm, and $\overline{AB} \cong \overline{AC}$

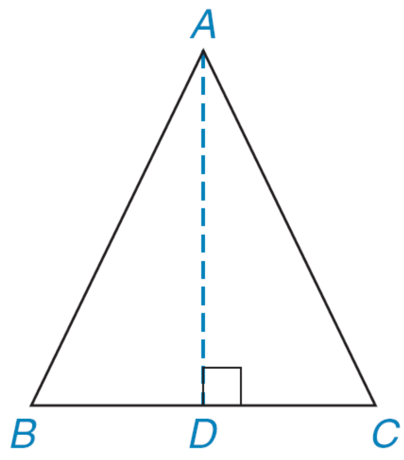


Figure 8.17

Solution:

a) $P_{ABC} = AB + AC + BC$

$= 5 + 6 + 7$

$= 18$ in.

Example 1 – *Solution*

cont'd

b) With $\overline{AB} \cong \overline{AC}$, $\triangle ABC$ is isosceles.

Then $\overline{AD}$ is the $\perp$ bisector of $\overline{BC}$.

If $BC = 6$, it follows that $DC = 3$.

Using the Pythagorean Theorem, we have

$$(AD)^2 + (DC)^2 = (AC)^2$$

$$8^2 + 3^2 = (AC)^2$$

Example 1 – *Solution*

cont'd

$$64 + 9 = (AC)^2$$

$$AC = \sqrt{73}$$

Now $P_{ABC} = 6 + \sqrt{73} + \sqrt{73} = 6 + 2\sqrt{73} \approx 23.1$ cm.

Note: Because $x + x = 2x$, we have $\sqrt{73} + \sqrt{73} = 2\sqrt{73}$.

HERON’S FORMULA

Heron's Formula

If the lengths of the sides of a triangle are known, the formula generally used to calculate the area is **Heron's Formula.**

One of the numbers found in this formula is the *semiperimeter* of a triangle, which is defined as one-half the perimeter.

For the triangle that has sides of lengths a, b, and c, the semiperimeter is $s = \frac{1}{2}(a + b + c)$.

Heron's Formula

Theorem 8.2.1 (Heron's Formula)

If the three sides of a triangle have lengths *a*, *b*, and *c*, then the area *A* of the triangle is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where the semiperimeter of the triangle is

$$s = \frac{1}{2}(a + b + c)$$

Example 3

Find the area of a triangle which has sides of lengths 4, 13, and 15. (See Figure 8.19.)

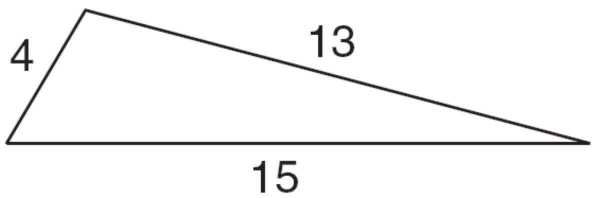


Figure 8.19

Solution:

If we designate the sides as $a = 4$, $b = 13$, and $c = 15$, the semiperimeter of the triangle is given by

$$s = \tfrac{1}{2}(4 + 13 + 15)$$

$$= \tfrac{1}{2}(32)$$

$$= 16$$

Example 3 – *Solution*

Therefore,

$$
\begin{aligned}
A &= \sqrt{s(s-a)(s-b)(s-c)} \\
&= \sqrt{16(16-4)(16-13)(16-15)} \\
&= \sqrt{16(12)(3)(1)} \\
&= \sqrt{576} \\
&= 24 \text{ units}^2
\end{aligned}
$$

Heron’s Formula

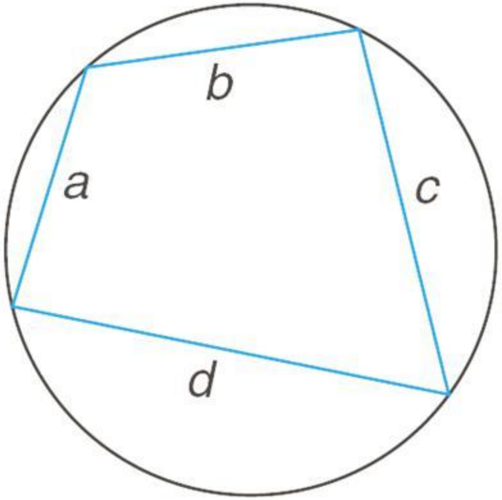
When the lengths of the sides of a quadrilateral are known, we can apply Heron’s Formula to find the area if the length of a diagonal is also known.

Heron's Formula

Theorem 8.2.2 (Brahmagupta's Formula)

For a cyclic quadrilateral with sides of lengths *a*, *b*, *c*, and *d*, the area is given by

$$A = \sqrt{(s - a)(s - b)(s - c)(s - d)},$$



where

$$s = \tfrac{1}{2}(a + b + c + d)$$

AREA OF A TRAPEZOID

Area of a Trapezoid

We know that the two parallel sides of a trapezoid are its *bases*.

The *altitude* is any line segment that is drawn perpendicular from one base to the other.

In Figure 8.21, $\overline{AB} \parallel \overline{DC}$ so and $\overline{AB}$ and $\overline{DC}$ are bases and $\overline{AE}$ is an altitude for the trapezoid.

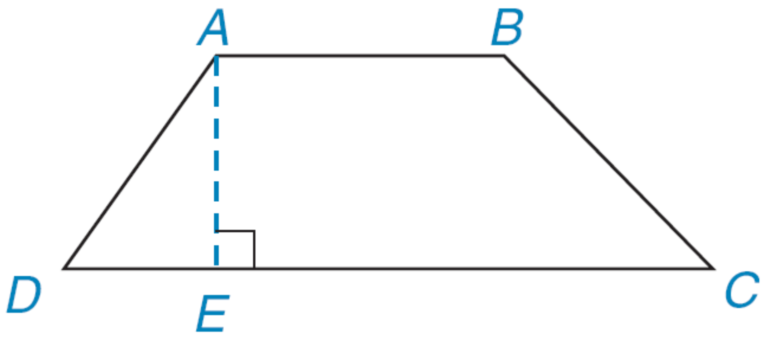


Figure 8.21

Area of a Trapezoid

We use the more common formula for the area of a triangle (namely, $A = \frac{1}{2}bh$) to develop our remaining theorems.

Theorem 8.2.3

The area A of a trapezoid whose bases have lengths b_1 and b_2 and whose altitude has length h is given by

$$A = \frac{1}{2}h(b_1 + b_2)$$

Example 5

Given that $\overline{RS} \parallel \overline{VT}$, find the area of the trapezoid in Figure 8.23. Note that RS = 5, TV = 13, and RW = 6.

Solution:

Let $RS = 5 = b_1$ and $TV = 13 = b_2$.

Also, $RW = h = 6$.

Now,

$$A = \frac{1}{2}h(b_1 + b_2)$$

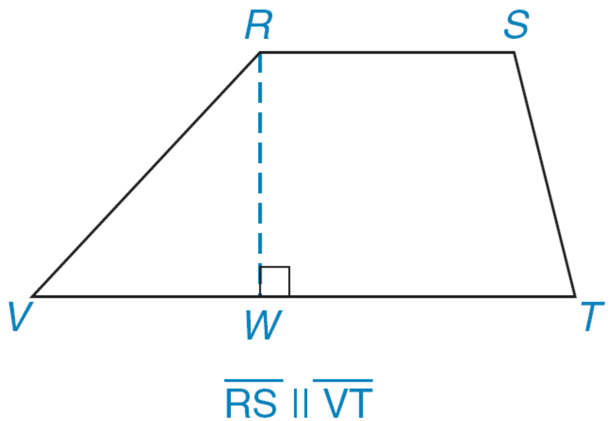


Figure 8.23

Example 5 – *Solution*

cont'd

becomes

$$A = \tfrac{1}{2} \cdot 6(5 + 13)$$

$$A = \tfrac{1}{2} \cdot 6 \cdot 18$$

$$= 3 \cdot 18$$

$$= 54 \text{ units}^2$$

QUADRILATERALS WITH PERPENDICULAR DIAGONALS

Quadrilaterals with Perpendicular Diagonals

Theorem 8.2.4

The area of any quadrilateral with perpendicular diagonals of lengths d_1 and d_2 is given by

$$A = \frac{1}{2}d_1d_2$$

AREA OF A RHOMBUS

Area of a Rhombus

We know that a rhombus is a parallelogram with two congruent adjacent sides. Among the properties of the rhombus, we proved “The diagonals of a rhombus are perpendicular.”

Thus, we have the following corollary of Theorem 8.2.4. See Figure 8.25.

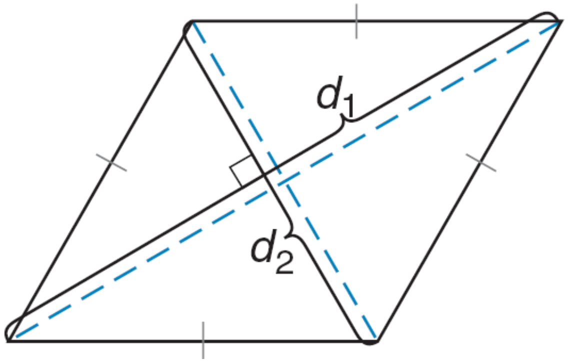


Figure 8.25

Area of a Rhombus

Corollary 8.2.5

The area A of a rhombus whose diagonals have lengths d_1 and d_2 is given by

$$A = \frac{1}{2}d_1d_2$$

Example 6 illustrates Corollary 8.2.5.

Example 6

Find the area of the rhombus *MNPQ* in Figure 8.26 if $MP = 12$ and $NQ = 16$.

Solution:

Applying Corollary 8.2.5,

$$A_{MNPQ} = \frac{1}{2}d_1d_2$$

$$= \frac{1}{2} \cdot 12 \cdot 16$$

$$= 96 \text{ units}^2$$

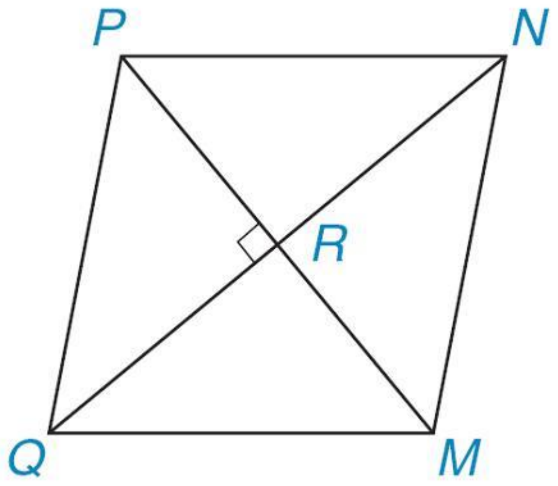


Figure 8.26

Area of a Rhombus

In problems involving the rhombus, we often utilize the fact that its diagonals are perpendicular bisectors of each other.

If the length of a side and the length of either diagonal are known, the length of the other diagonal can be found by applying the Pythagorean Theorem.

AREA OF A KITE

Area of a Kite

For a kite, one diagonal is the perpendicular bisector of the other. (See Figure 8.27.)

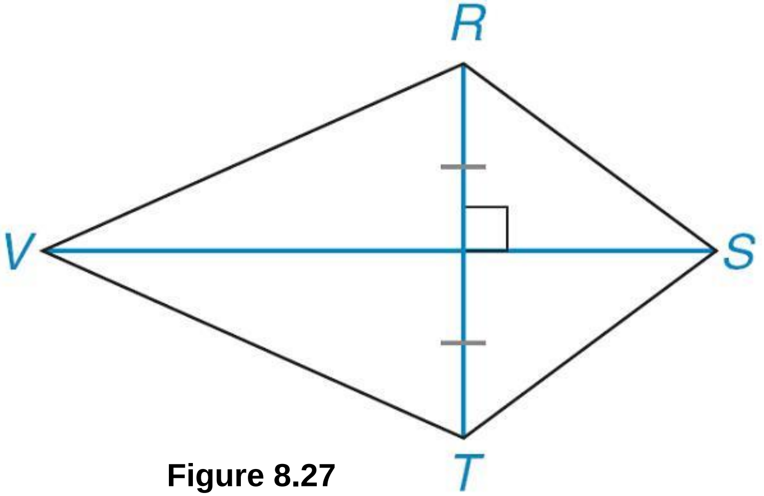


Figure 8.27

Corollary 8.2.6

The area A of a kite whose diagonals have lengths d_1 and d_2 is given by

$$A = \frac{1}{2}d_1d_2$$

Example 7

Find the length of $\overline{RT}$ in Figure 8.28 if the area of the kite *RSTV* is 360 in^2 and $SV = 30$ in.

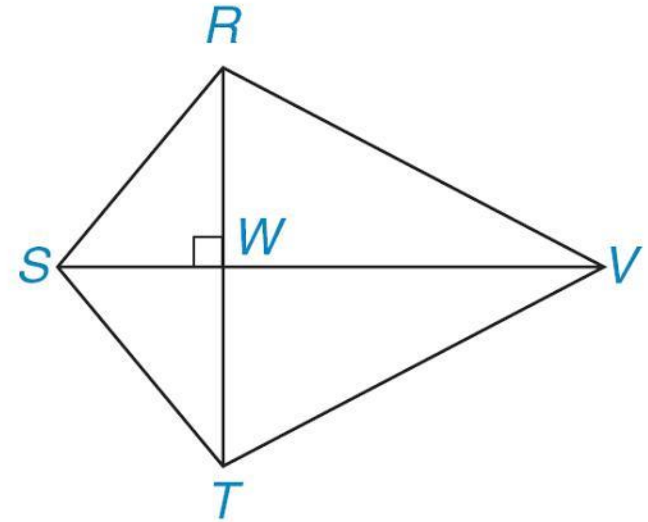


Figure 8.28

Solution:

$A = \frac{1}{2}d_1d_2$ becomes $360 = \frac{1}{2}(30)d$, in which d is the length of the remaining diagonal $\overline{RT}$.

Then $360 = 15d$, which means that $d = 24$.

Then $RT = 24$ in.

AREAS OF SIMILAR POLYGONS

Areas of Similar Polygons

The following theorem compares the areas of similar triangles. In Figure 8.29, we refer to the areas of the similar triangles as A_1 and A_2.

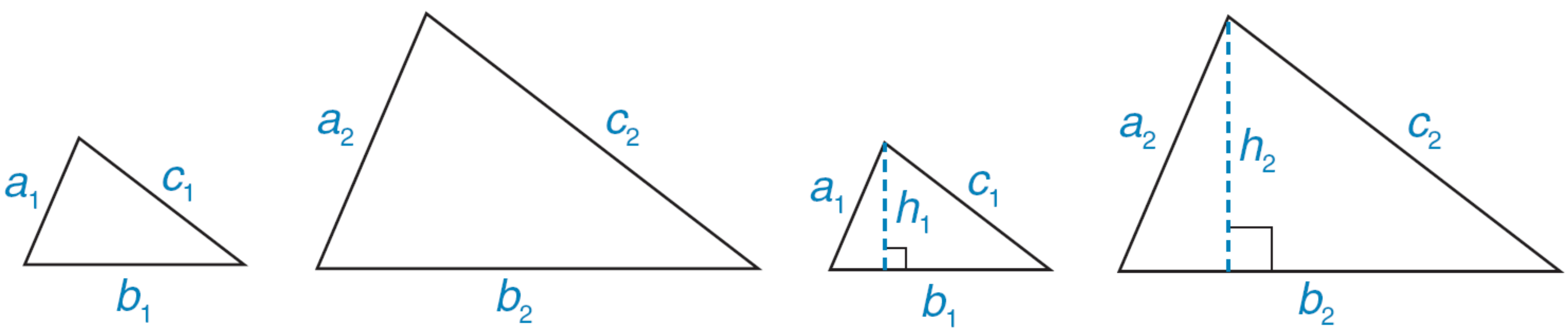


Figure 8.29

Areas of Similar Polygons

The triangle with area A_1 has sides of lengths a_1, b_1, and c_1, and the triangle with area A_2 has sides of lengths a_2, b_2, and c_2.

Where a_1 corresponds to a_2, b_1 to b_2, and c_1 to c_2, Theorem 8.2.7 implies that

$$\frac{A_1}{A_2} = \left(\frac{a_1}{a_2}\right)^2 \quad \text{or} \quad \frac{A_1}{A_2} = \left(\frac{b_1}{b_2}\right)^2 \quad \text{or} \quad \frac{A_1}{A_2} = \left(\frac{c_1}{c_2}\right)^2$$

Areas of Similar Polygons

Theorem 8.2.7

The ratio of the areas of two similar triangles equals the square of the ratio of the lengths of any two corresponding sides; that is,

$$\frac{A_1}{A_2} = \left(\frac{a_1}{a_2}\right)^2$$

Example 8

Use the ratio $\frac{A_1}{A_2}$ to compare the areas of

a) two similar triangles in which the sides of the first triangle are $\frac{1}{2}$ as long as the sides of the second triangle.

b) two squares in which each side of the first square is 3 times as long as each side of the second square.

Example 8(a) – *Solution*

$$s_1 = \frac{1}{2} s_2,$$

so $\frac{s_1}{s_2} = \frac{1}{2}$. (See Figure 8.30.)

Now $\frac{A_1}{A_2} = \left(\frac{s_1}{s_2}\right)^2$,

so that $\frac{A_1}{A_2} = \left(\frac{1}{2}\right)^2$ or $\frac{A_1}{A_2} = \frac{1}{4}$.

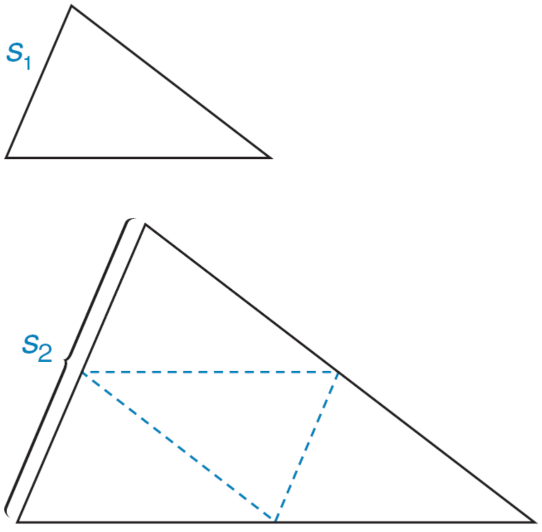


Figure 8.30

That is, the area of the first triangle is $\frac{1}{4}$ the area of the second triangle.

Example 8(b) – *Solution*

cont'd

$s_1 = 3s_2$, so $\frac{s_1}{s_2} = 3$. (See Figure 8.31.)

$\frac{A_1}{A_2} = \left(\frac{s_1}{s_2}\right)^2$, so that $\frac{A_1}{A_2} = (3)^2$ or $\frac{A_1}{A_2} = 9$.

That is, the area of the first square is 9 times the area of the second square.

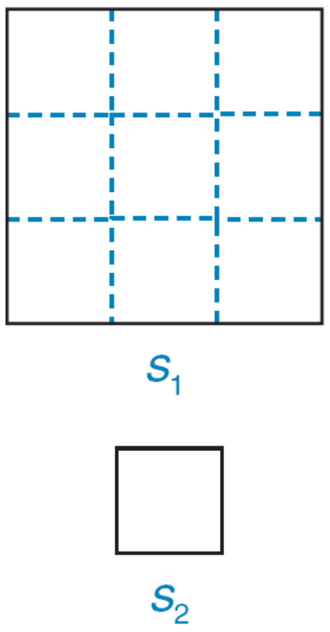


Figure 8.31