Chapter 8

Areas of Polygons and Circles

8.1 Area and Initial Postulates

Area and Initial Postulates

Because lines are *one-dimensional*, we consider only length when measuring a line segment. A line segment is measured in linear units such as inches, centimeters, or yards.

When a line segment measures 5 centimeters, we write $AB = 5$ cm (or $AB = 5$ if units are not stated). The instrument for measuring length is the ruler.

Area and Initial Postulates

A plane is an infinite *two-dimensional* surface. A closed or bounded portion of the plane is called a **region**. When a region such as *R* in plane *M* [see Figure 8.1(a)] is measured, we call this measure the "area of the plane region."

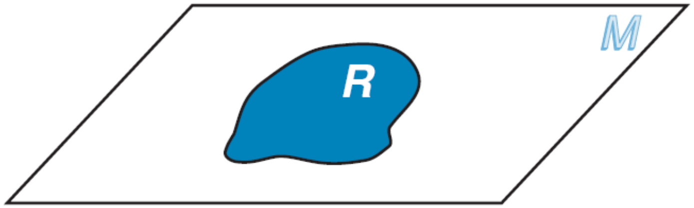


Figure 8.1(a)

Area and Initial Postulates

The unit used to measure area is known a **square unit** because it is a square with each side of length 1 [see Figure 8.1(b)].

The measure of the area of region *R* is the number of nonoverlapping square units that can be placed adjacent to each other in the region.

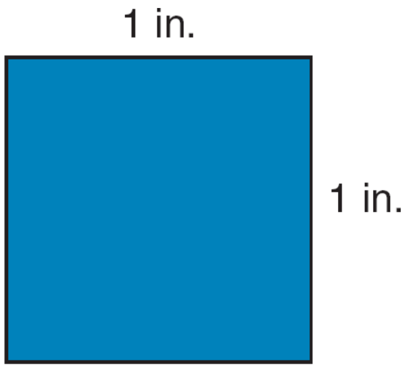


Figure 8.1(b)

Square units (not linear units) are used to measure area. Using an exponent, we write square inches as in^2. The unit represented by Figure 8.1(b) is 1 square inch or 1 in^2.

Area and Initial Postulates

In Figure 8.2, the regions have measurable areas and are bounded by figures encountered in earlier chapters.

A region is **bounded** if we can distinguish between its interior and its exterior; in calculating area, we measure the interior of the region.

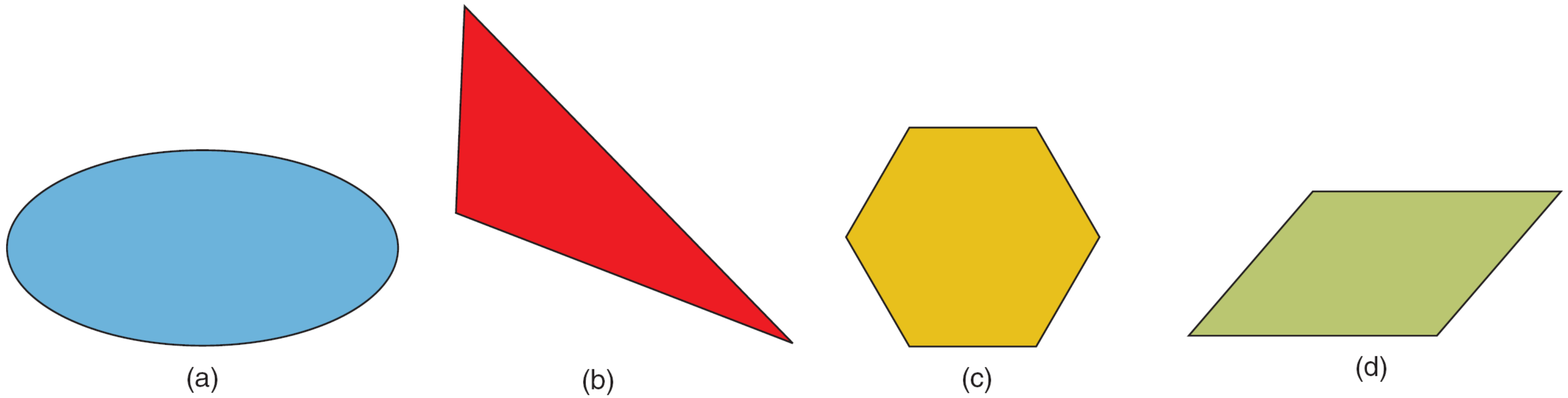


Figure 8.2

Area and Initial Postulates

We can measure the area of the region within a triangle [see Figure 8.2(b)]. However, we cannot actually measure the area of the triangle itself (three line segments do not have area). Nonetheless, the area of the region within a triangle is commonly referred to as the *area of the triangle.*

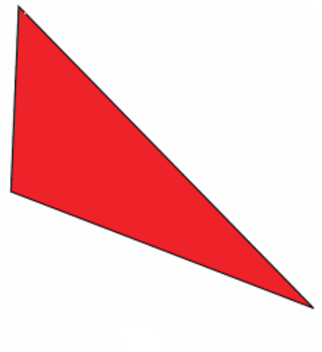


Figure 8.2(b)

The preceding discussion does not formally define a region or its area. These are accepted as the undefined terms in the following postulate.

Area and Initial Postulates

Postulate 18 (Area Postulate)
Corresponding to every bounded region is a unique positive number *A*, known as the area of that region.

One way to estimate the area of a region is to place it in a grid, as shown in Figure 8.3.

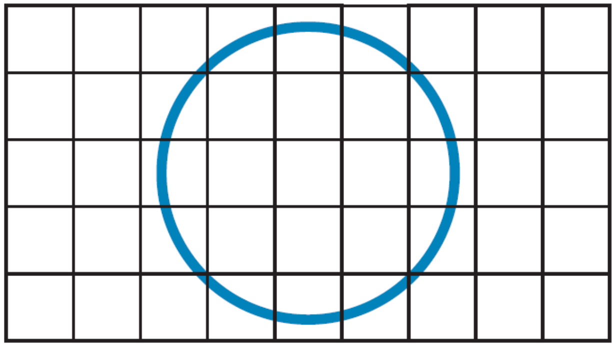


Figure 8.3

Area and Initial Postulates

Counting only the number of whole squares inside the region gives an approximation that is less than the actual area.

On the other hand, counting squares that are inside or partially inside provides an approximation that is greater than the actual area.

A fair estimate of the area of a region is often given by the average of the smaller and larger approximations just described.

Area and Initial Postulates

If the area of the circle shown in Figure 8.3 is between 9 and 21 square units, we might estimate its area to be $\frac{9 + 21}{2}$ or 15 square units.

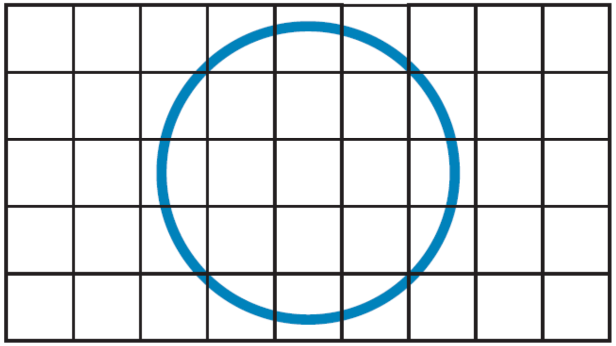


Figure 8.3

Area and Initial Postulates

To explore another property of area, we consider $\triangle ABC$ and $\triangle DEF$ (which are congruent) in Figure 8.4. One triangle can be placed over the other so that they coincide. How are the areas of the two triangles related? The answer is found in the following postulate.

Figure 8.4

Area and Initial Postulates

Postulate 19

If two closed plane figures are congruent, then their areas are equal.

Example 1

In Figure 8.5, points *B* and *C* trisect $\overline{AD}$; $\overline{EC} \perp \overline{AD}$. Name two triangles with equal areas.

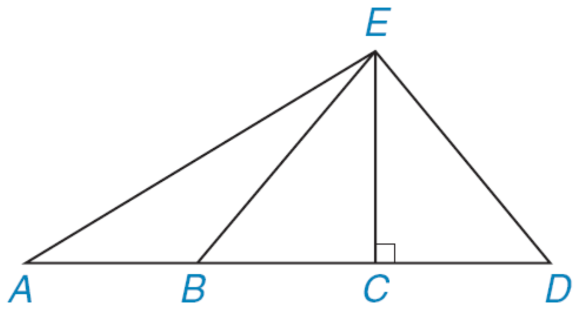


Figure 8.5

Solution:

$\triangle ECB \cong \triangle ECD$ by SAS.

Then $\triangle ECB$ and $\triangle ECD$ have equal areas according to Postulate 19.

Area and Initial Postulates

Consider Figure 8.6. The entire region is bounded by a curve and then subdivided by a line segment into smaller regions *R* and *S*. These regions have a common boundary and do not overlap.

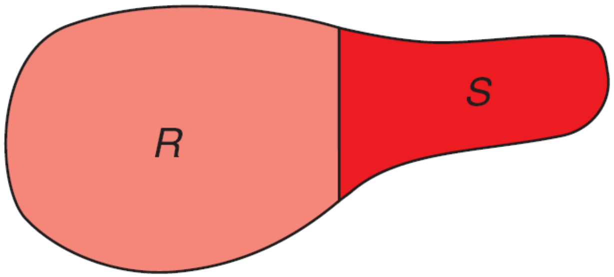


Figure 8.6

Area and Initial Postulates

Because a numerical area can be associated with each region R and S, the area of $R \cup S$ (read as "R union S" and meaning region R joined to region S) is equal to the sum of the areas of R and S. This leads to Postulate 20, in which A_R represents the "area of region R," A_S represents the "area of region S," and $A_{R \cup S}$ represents the "area of region $R \cup S$."

Postulate 20 (Area-Addition Postulate)

Let R and S be two enclosed regions that do not overlap. Then $A_{R \cup S} = A_R + A_S$

AREA OF A RECTANGLE

Area of a Rectangle

Study rectangle *MNPQ* in Figure 8.8, and note that it has dimensions of 3 cm and 4 cm. The number of squares, 1 cm on a side, in the rectangle is 12.

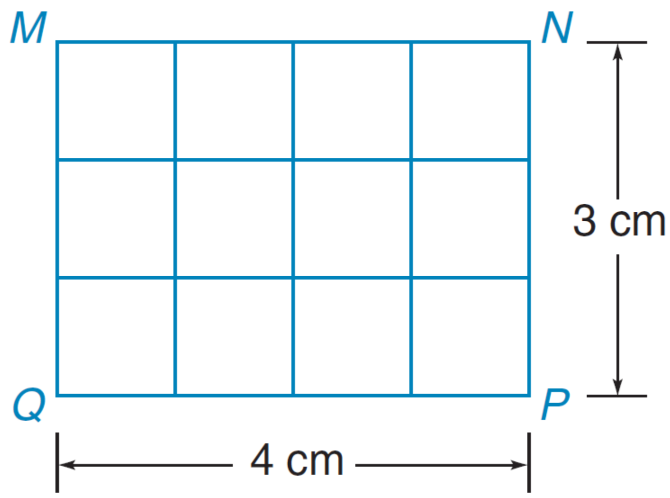


Figure 8.8

Area of a Rectangle

The unit of area is cm^2. Multiplication of dimensions is handled like algebraic multiplication.

Compare

$3x \cdot 4x = 12x^2$ and $3 \text{ cm} \cdot 4 \text{ cm} = 12 \text{ cm}^2$

If the units used to measure the dimensions of a region are *not* the same, then they must be converted into like units in order to calculate area.

Area of a Rectangle

For instance, if we need to multiply 2 ft by 6 in., we note that 2 ft = 2(12 in.) = 24 in., so A = 2 ft • 6 in.= 24 in. • 6 in., and A = 144 in^2.

Alternatively, $6 \text{ in.} = 6\left(\frac{1}{12}\text{ ft}\right) = \frac{1}{2}\text{ ft, so } A = 2\text{ ft} \cdot \frac{1}{2}\text{ ft} = 1\,\text{ft}^2.$ Because the area is unique, we know that 1 ft^2 = 144 in^2. See Figure 8.9.

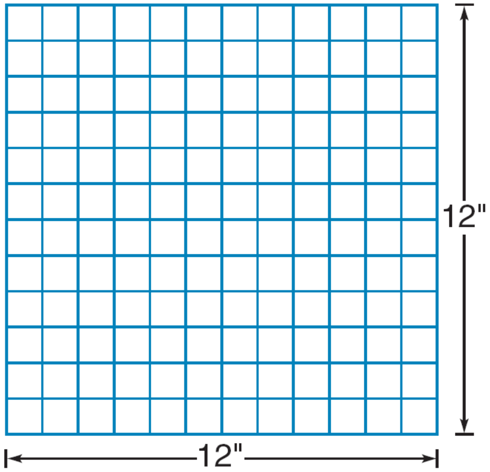


Figure 8.9

Area of a Rectangle

Recall that one side of a rectangle is called its *base* and that any line segment between sides and perpendicular to the base is called the *altitude* of the rectangle.

Area of a Rectangle

Postulate 21

The area A of a rectangle whose base has length b and whose altitude has length h is given by $A = bh$.

It is also common to describe the dimensions of a rectangle as length ℓ and width w. The area of the rectangle is then written $A = \ell w$.

Example 3

Find the area of rectangle *ABCD* in Figure 8.10 if *AB* = 12 cm and *AD* = 7 cm.

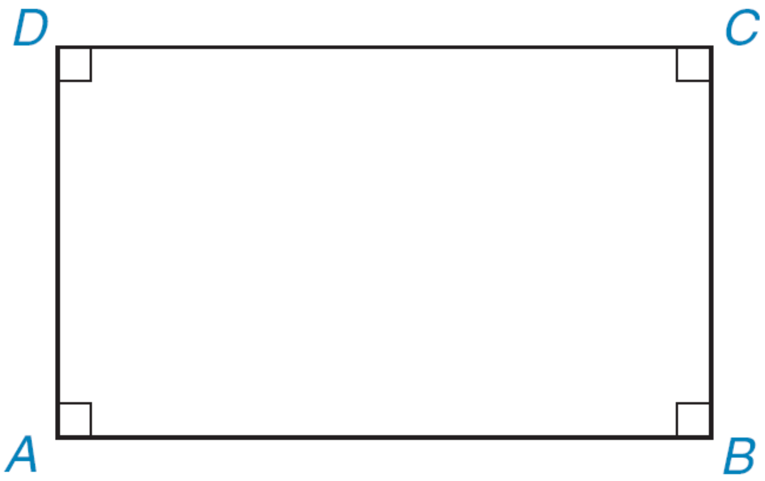


Figure 8.10

Example 3 – *Solution*

Because it makes no difference which dimension is chosen as base b and which as altitude h, we arbitrarily choose $AB = b = 12$ cm and $AD = h = 7$ cm.

Then

$$\begin{aligned} A &= bh \\ &= 12 \text{ cm} \cdot 7 \text{ cm} \\ &= 84 \text{ cm}^2 \end{aligned}$$

If units are not provided for the dimensions of a region, we assume that they are alike. In such a case, we simply give the area as a number of square units.

Area of a Rectangle

Theorem 8.1.1

The area A of a square whose sides are each of length s is given by $A = s^2$.

AREA OF A PARALLELOGRAM

Area of a Parallelogram

A rectangle's altitude is one of its sides, but that is not true of a parallelogram.

An **altitude** of a parallelogram is a perpendicular line segment drawn from one side to the opposite side, known as the **base**.

A side may have to be extended in order to show this altitude-base relationship in a drawing.

Area of a Parallelogram

In Figure 8.11(a), if $\overline{RS}$ is designated as the base, then any of the segments $\overline{ZR}$, $\overline{VX}$, or $\overline{YS}$ is an altitude corresponding to that base (or, for that matter, to base $\overline{VT}$).

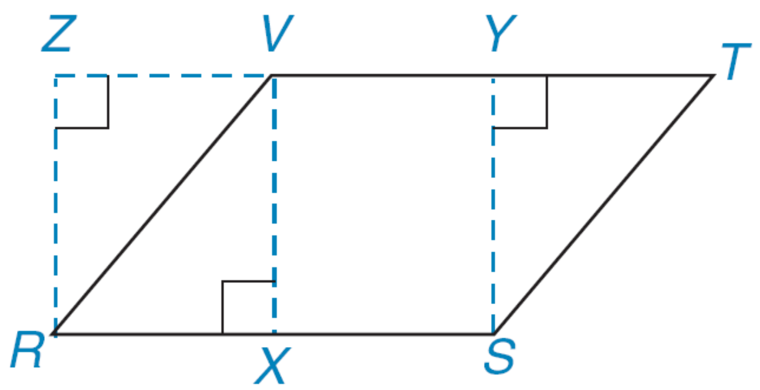


Figure 8.11(a)

Area of a Parallelogram

Another look at $\square RSTV$ [in Figure 8.11(b)] shows that $\overline{ST}$ (or $\overline{VR}$) could just as well have been chosen as the base. Possible choices for the corresponding altitude in this case include $\overline{VH}$ and $\overline{SG}$.

In the theorem that follows, it is necessary to select a base and an altitude drawn to that base!

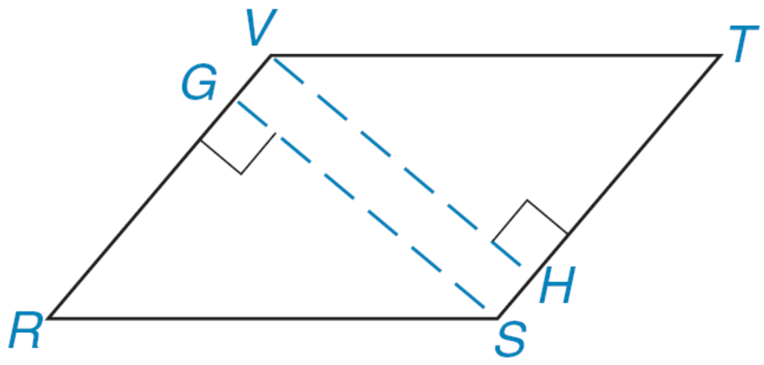


Figure 8.11(b)

Area of a Parallelogram

Theorem 8.1.2

The area A of a parallelogram with a base of length b and with corresponding altitude of length h is given by

$$A = bh$$

Example 4

Given that all dimensions in Figure 8.13 are in inches, find the area of ▱$MNPQ$ by using base

a) MN.   b) PN.

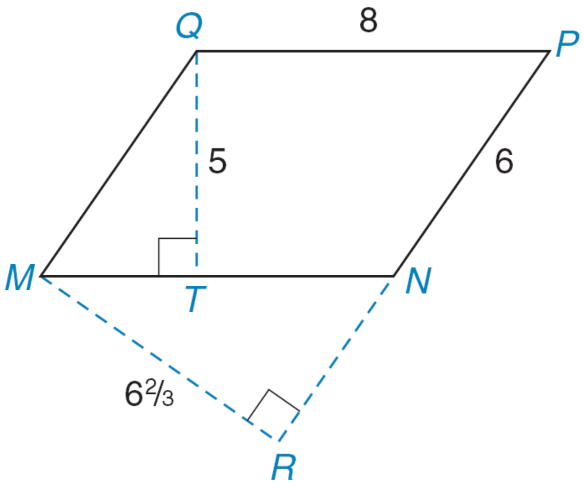


Figure 8.13

Example 4 – *Solution*

a) $MN = QP = b = 8$, and the corresponding altitude is of length $QT = h = 5$.

Then $A = 8 \cdot 5$

$= 40 \text{ in}^2$

b) $PN = b = 6$, so the corresponding altitude length is $MR = h = 6\frac{2}{3}$.

Then $A = 6 \cdot 6\frac{2}{3}$

$= 6 \cdot \frac{20}{3}$

$= 40 \text{ in}^2$

Area of a Parallelogram

In Example 4, the area of ▱*MNPQ* was not changed when a different base (length) and the length of its corresponding altitude were used to calculate its area.

Postulate 18 (Area Postulate)

Corresponding to every bounded region is a unique positive number *A*, known as the area of that region.

AREA OF A TRIANGLE

Area of a Triangle

The formula used to calculate the area of a triangle follows easily from the formula for the area of a parallelogram. In the formula, any side of the triangle can be chosen as its base; however, we must use the length of the corresponding altitude for that base.

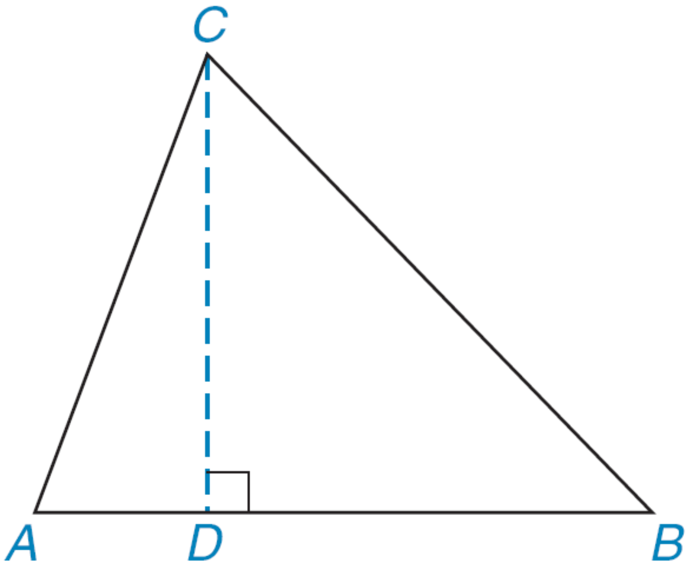
Theorem 8.1.3

The area A of a triangle whose base has length b and whose corresponding altitude has length h is given by

$$A = \frac{1}{2}bh$$

Example 6

In the figure, find the area of $\triangle ABC$ if AB = 10 cm and CD = 7 cm.



Example 6 – *Solution*

With $\overline{AB}$ as base, b = 10 cm. The corresponding altitude for base $\overline{AB}$ is $\overline{CD}$, so h = 7 cm.

Now $$A = \frac{1}{2}bh$$

becomes $$A = \frac{1}{2} \cdot 10\text{ cm} \cdot 7\text{ cm}$$

$$A = 35\text{ cm}^2$$

Area of a Parallelogram

Corollary 8.1.4

The area of a right triangle with legs of lengths *a* and *b* is given by

$$A = \frac{1}{2}ab.$$