Chapter 7

Locus and Concurrence

7.2 Concurrence of Lines

Concurrence of Lines

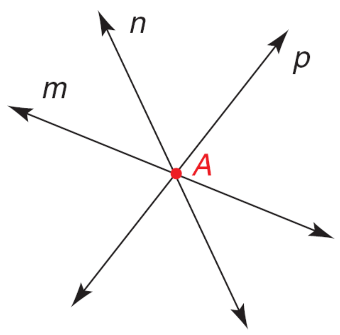
In this section, we consider coplanar lines that share a common point. However, any group of lines in space that have a single point in common are also known as *concurrent* lines.

Definition

A number of lines are **concurrent** if they have exactly one point in common.

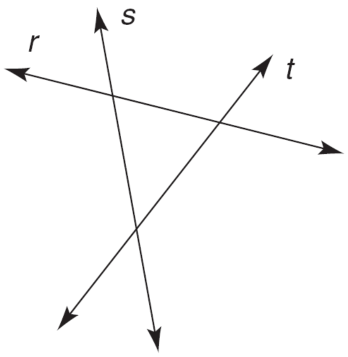
Concurrence of Lines

The three lines in Figure 7.14 are concurrent at point *A*. The three lines in Figure 7.15 are not concurrent even though any pair of lines (such as *r* and *s*) do intersect.



m, *n*, and *p* *are* concurrent

Figure 7.14



r, *s*, and *t* are *not* concurrent

Figure 7.15

Parts of lines (rays or segments) are concurrent if they are parts of concurrent lines and the parts share a common point.

Concurrence of Lines

Theorem 7.2.1

The three bisectors of the angles of a triangle are concurrent.

Example 1

Give an informal proof of Theorem 7.2.1.

Proof:

In Figure 7.16(a), the bisectors of $\angle BAC$ and $\angle ABC$ intersect at point E.

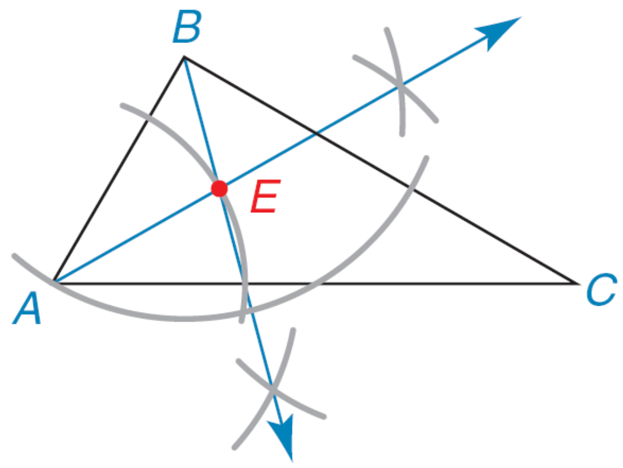


Figure 7.16(a)

Example 1

cont'd

Because the bisector of $\angle BAC$ is the locus of points equidistant from the sides of $\angle BAC$, we know that $\overline{EM} \cong \overline{EN}$ in Figure 7.16(b). Similarly, $\overline{EM} \cong \overline{EP}$ because E is on the bisector of $\angle ABC$.

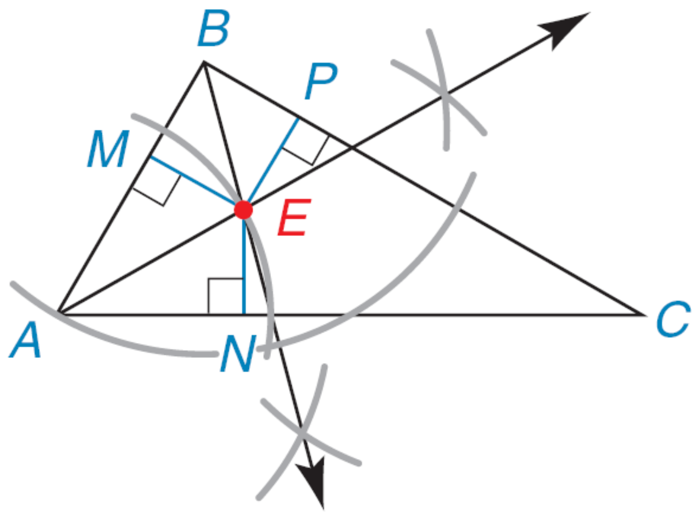


Figure 7.16(b)

Example 1

cont'd

By the Transitive Property of Congruence, it follows that $\overline{EP} \cong \overline{EN}$.

Because the bisector of an angle is the locus of points equidistant from the sides of the angle, *E* is also on the bisector of the third angle, $\angle ACB$.

Thus, the angle bisectors are concurrent at point *E*.

Concurrence of Lines

The point *E* at which the angle bisectors meet in Example 1 is the **incenter** of the triangle.

Theorem 7.2.2

The three perpendicular bisectors of the sides of a triangle are concurrent.

The point at which the perpendicular bisectors of the sides of a triangle meet is the **circumcenter** *F* of the triangle. The term *circumcenter* is easily remembered as the *center* of the *circum*scribed circle.

Concurrence of Lines

The incenter and the circumcenter of a triangle are generally distinct points. However, it is possible for the two centers to coincide in a special type of triangle.

Although the incenter of a triangle always lies in the interior of the triangle, the circumcenter of an obtuse triangle will lie in the exterior of the triangle. See Figure 7.19.

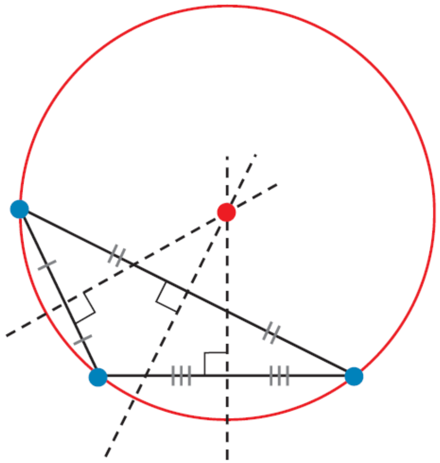


Figure 7.19

Concurrence of Lines

To complete the discussion of concurrence, we include a theorem involving the altitudes of a triangle and a theorem involving the medians of a triangle.

Theorem 7.2.3

The three altitudes of a triangle are concurrent.

Concurrence of Lines

The point of concurrence for the three altitudes of a triangle is the **orthocenter** of the triangle. In Figure 7.20(a), point *N* is the orthocenter of $\triangle DEF$.

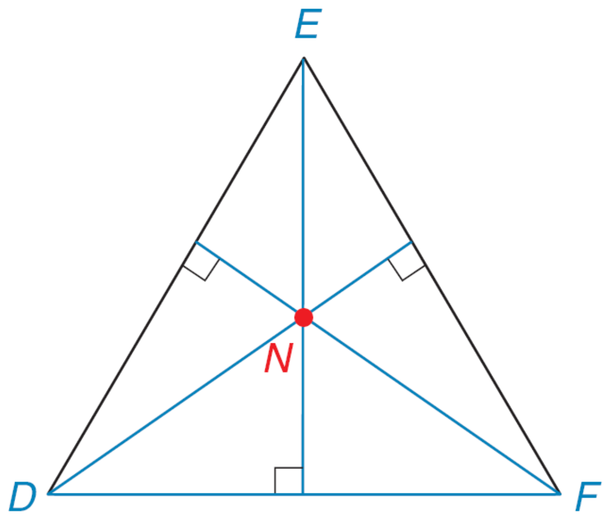


Figure 7.20(a)

Concurrence of Lines

For the obtuse triangle in Figure 7.20(b), we see that orthocenter X lies in the exterior of $\triangle RST$.

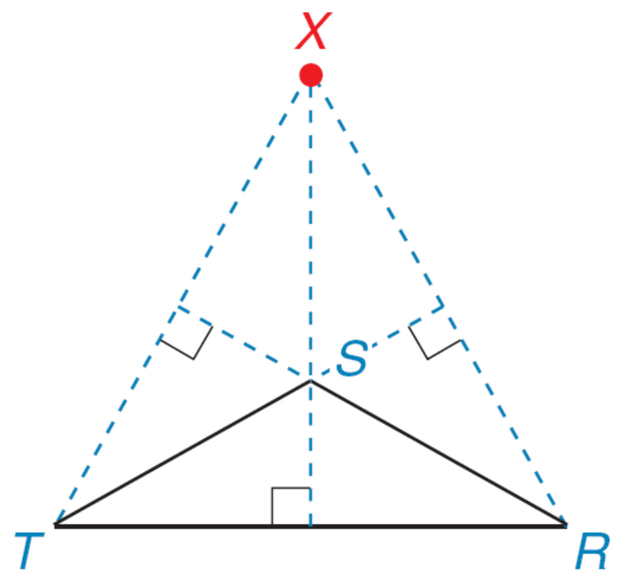


Figure 7.20(b)

Concurrence of Lines

Theorem 7.2.4

The three medians of a triangle are concurrent at a point that is two-thirds the distance from any vertex to the midpoint of the opposite side.

Concurrence of Lines

The point of concurrence *C* for the three medians is the **centroid** of the triangle in Figure 7.23. *M*, *N*, and *P* are midpoints of the sides of $\triangle RST$.

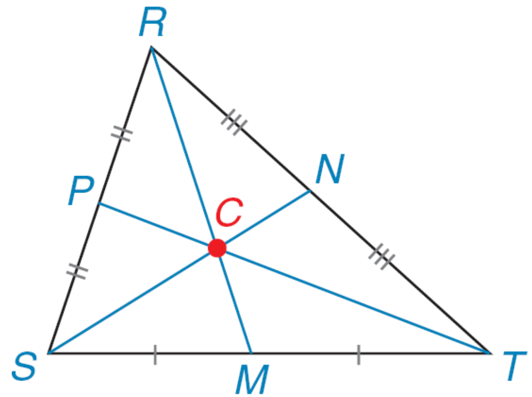


Figure 7.23

According to Theorem 7.2.4,

$$RC = \frac{2}{3}(RM),\ SC = \frac{2}{3}(SN),\ \text{and}\ TC = \frac{2}{3}(TP).$$

Concurrence of Lines

Based upon the figure 7.23, the following table shows several interpretations of Theorem 7.2.4

TABLE 7.1 **Centroid C in $\triangle RST$**

$RC = \frac{2}{3}(RM)$	$CM = \frac{1}{3}(RM)$	$RC = 2(CM)$	$CM = \frac{1}{2}(RC)$
$SC = \frac{2}{3}(SN)$	$CN = \frac{1}{3}(SN)$	$SC = 2(CN)$	$CN = \frac{1}{2}(SC)$
$TC = \frac{2}{3}(TP)$	$CP = \frac{1}{3}(TP)$	$TC = 2(CP)$	$CP = \frac{1}{2}(TC)$

Concurrence of Lines

The centroid of a triangular region is sometimes called its *center of mass* or *center of gravity*. This is because the region of uniform thickness “balances” upon the point known as its centroid.

It is *possible* for the angle bisectors of certain quadrilaterals to be concurrent.

Likewise, the perpendicular bisectors of the sides of a quadrilateral *can* be concurrent. Of course, there are four angle bisectors and four perpendicular bisectors of sides to consider.