Chapter 6 Circles

6.2 More Angle Measures in the Circle

More Angle Measures in the Circle

We begin this section by considering lines, rays, and segments that are related to the circle. We assume that the lines and circles are coplanar.

Definition

A **tangent** is a line that intersects a circle at exactly one point; the point of intersection is the **point of contact,** or **point of tangency.**

The term *tangent* also applies to a line segment or ray that is part of a tangent line to a circle. In each case, the tangent touches the circle at one point.

More Angle Measures in the Circle

Definition

A **secant** is a line (or segment or ray) that intersects a circle at exactly two points.

In Figure 6.21(a), line *s* is a secant to $\odot O$; also, line *t* is a tangent to $\odot O$ and point *C* is its point of contact.

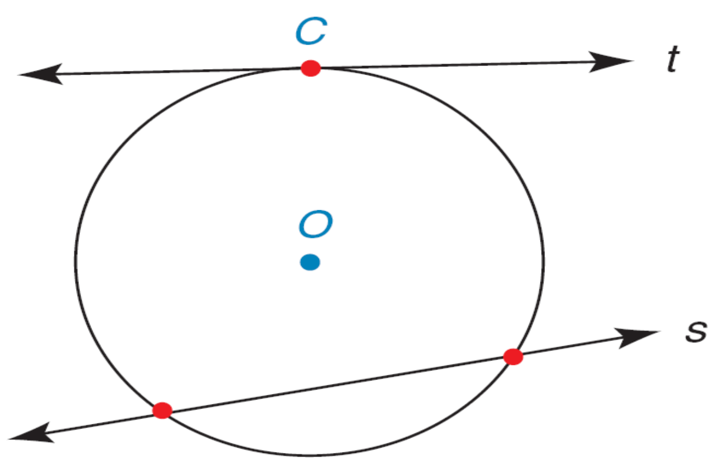


Figure 6.21(a)

More Angle Measures in the Circle

In Figure 6.21(b), $\overline{AB}$ is a tangent to $\odot Q$ and point T is its point of tangency; $\overrightarrow{CD}$ is a secant with points of intersection at E and F.

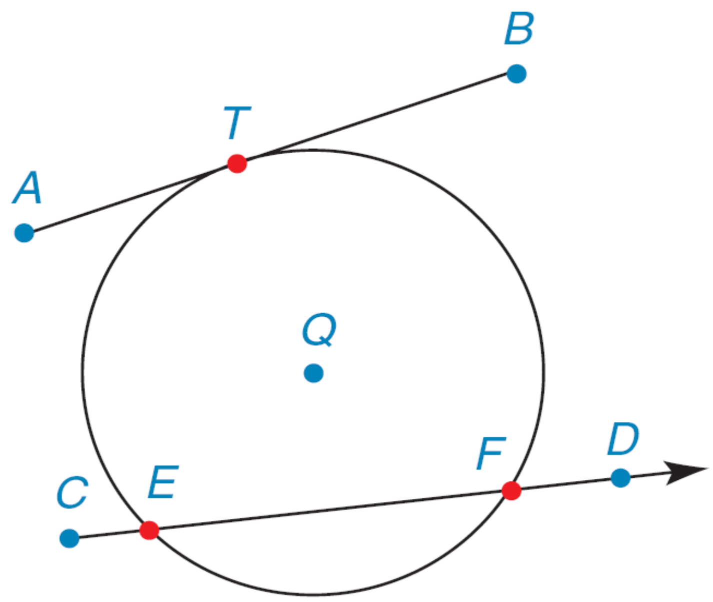


Figure 6.21(b)

More Angle Measures in the Circle

Definition

A polygon is **inscribed in a circle** if its vertices are points on the circle and its sides are chords of the circle. Equivalently, the circle is said to be **circumscribed about the polygon.** The polygon inscribed in a circle is further described as a **cyclic polygon.**

More Angle Measures in the Circle

In Figure 6.22, $\triangle ABC$ is inscribed in $\odot O$ and quadrilateral $RSTV$ is inscribed in $\odot Q$.

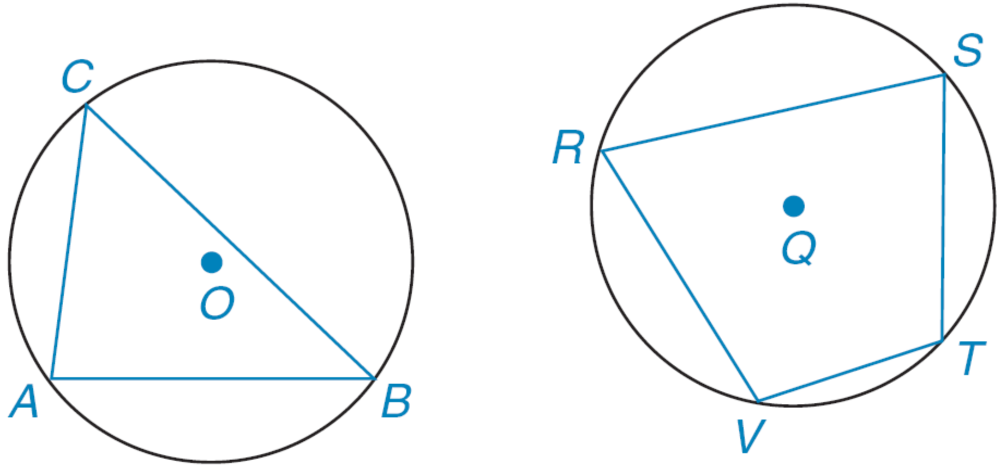


Figure 6.22

Conversely, $\odot O$ is circumscribed about $\triangle ABC$ and $\odot Q$ is circumscribed about quadrilateral $RSTV$.

More Angle Measures in the Circle

Note that $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are chords of $\odot O$ and that $\overline{RS}$, $\overline{ST}$, $\overline{TV}$, and $\overline{RV}$ are chords of $\odot Q$.

$\triangle ABC$ and quadrilateral *RSTV* are cyclic polygons.

Theorem 6.2.1

If a quadrilateral is inscribed in a circle, the opposite angles are supplementary.

Alternative Form: The opposite angles of a cyclic quadrilateral are supplementary.

More Angle Measures in the Circle

Definition

A polygon is **circumscribed about a circle** if all sides of the polygon are line segments tangent to the circle; also, the circle is said to be **inscribed in the polygon.**

More Angle Measures in the Circle

In Figure 6.24(a), $\triangle ABC$ is circumscribed about $\odot D$.

In Figure 6.24(b), square $MNPQ$ is circumscribed about $\odot T$.

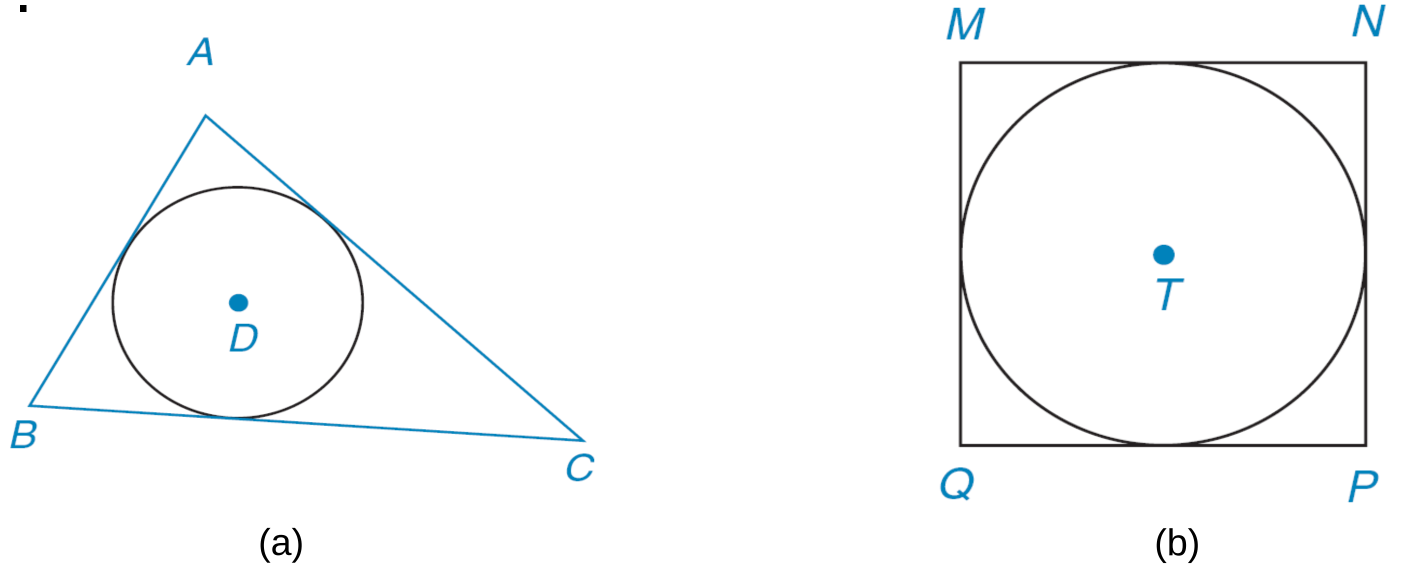


Figure 6.24

Furthermore, $\odot D$ is inscribed in $\triangle ABC$, and $\odot T$ is inscribed in square $MNPQ$.

More Angle Measures in the Circle

Note that $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ are tangents to $\odot D$ and that $\overline{MN}$, $\overline{NP}$, $\overline{PQ}$, and $\overline{MQ}$ are tangents to $\odot T$.

More Angle Measures in the Circle

We know that a central angle has a measure equal to the measure of its intercepted arc and that an inscribed angle has a measure equal to one-half the measure of its intercepted arc.

Now we consider another type of angle in the circle.

More Angle Measures in the Circle

Theorem 6.2.2

The measure of an angle formed by two chords that intersect within a circle is one-half the sum of the measures of the arcs intercepted by the angle and its vertical angle.

In Figure 6.25(a), $\angle 1$ and $\angle AEC$ are vertical angles; also $\angle 1$ intercepts $\overparen{DB}$ and $\angle AEC$ intercepts $\overparen{AC}$.

According to Theorem 6.2.2,

$$m\angle 1 = \frac{1}{2}(m\overparen{AC} + m\overparen{DB})$$

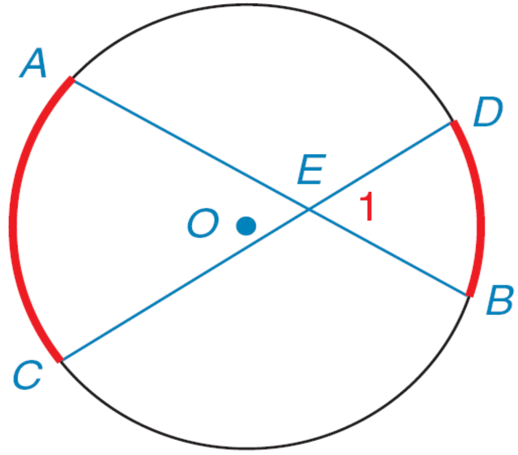


Figure 6.25(a)

Example 1

In Figure 6.25(a), $m\overset{\frown}{AC} = 84°$ and $m\overset{\frown}{DB} = 62°$. Find $m\angle 1$.

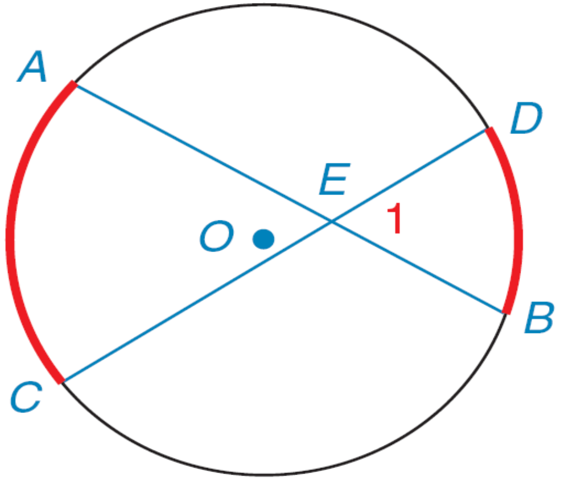


Figure 6.25(a)

Example 1 – *Solution*

By Theorem 6.2.2,

$$m\angle 1 = \frac{1}{2}(m\overset{\frown}{AC} + m\overset{\frown}{DB})$$

$$= \frac{1}{2}(84° + 62°)$$

$$= \frac{1}{2}(146°)$$

$$= 73°$$

More Angle Measures in the Circle

Recall that a circle separates points in the plane into three sets: points *in the interior* of the circle, points *on* the circle, and points *in the exterior* of the circle.

In Figure 6.26, point *A* and center *O* are in the **interior** of $\odot O$ because their distances from center *O* are less than the length of the radius.

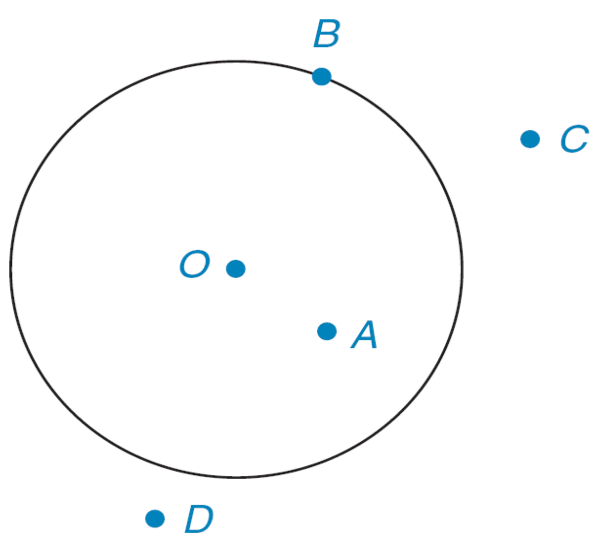


Figure 6.26

More Angle Measures in the Circle

Point *B* is on the circle, but points *C* and *D* are in the **exterior** of $\odot O$ because their distances from *O* are greater than the length of the radius.

In the proof of Theorem 6.2.3, we use the fact that a tangent to a circle cannot contain an interior point of the circle.

Theorem 6.2.3

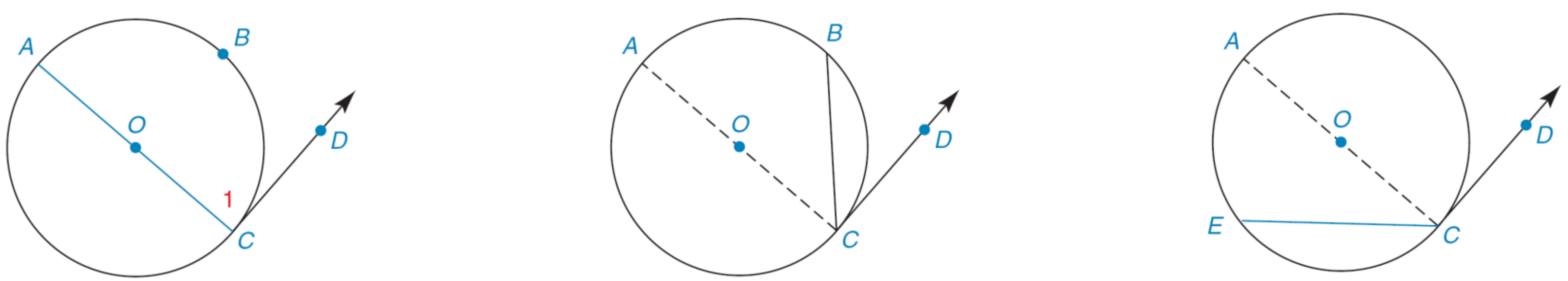
The radius (or any other line through the center of a circle) drawn to a tangent at the point of tangency is perpendicular to the tangent at that point.

More Angle Measures in the Circle

A consequence of Theorem 6.2.3 is Corollary 6.2.4, which has three possible cases, illustrated in Figure 6.29.

Corollary 6.2.4

The measure of an angle formed by a tangent and a chord drawn to the point of tangency is one-half the measure of the intercepted arc. (See Figure 6.29.)



(a) Case 1
The chord is a diameter.

(b) Case 2
The diameter is in the exterior of the angle.

(c) Case 3
The diameter lies in the interior of the angle.

Figure 6.29

More Angle Measures in the Circle

Theorem 6.2.5

The measure of an angle formed when two secants intersect at a point outside the circle is one-half the difference of the measures of the two intercepted arcs.

Theorems 6.2.5–6.2.7 show that any angle formed by two lines that intersect *outside* a circle has a measure equal to one-half of the difference of the measures of the two intercepted arcs.

More Angle Measures in the Circle

Theorem 6.2.6

If an angle is formed by a secant and a tangent that intersect in the exterior of a circle, then the measure of the angle is one-half the difference of the measures of its intercepted arcs.

According to Theorem 6.2.6,

$$m\angle L = \frac{1}{2}(m\overset{\frown}{HJ} - m\overset{\frown}{JK})$$

in Figure 6.33.

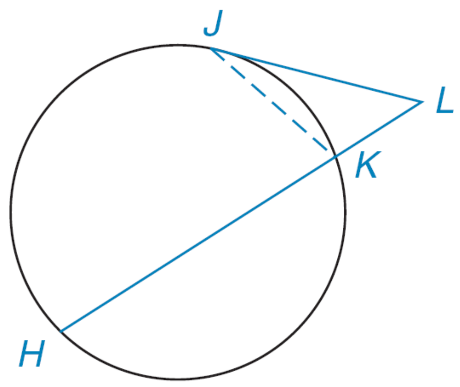


Figure 6.33

Again, we must subtract the measure of the smaller arc from the measure of the larger arc.

More Angle Measures in the Circle

A quick study of the figures that illustrate Theorems 6.2.5–6.2.7 shows that the smaller intercepted arc is "nearer" the vertex of the angle and that the larger arc is "farther from" the vertex of the angle.

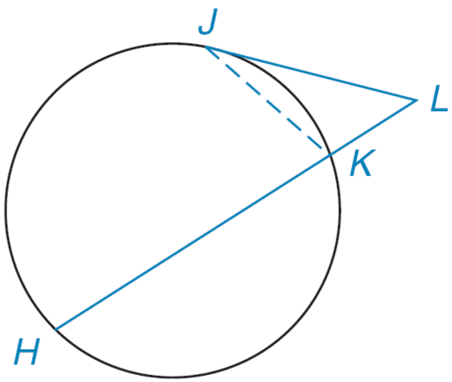


Figure 6.33

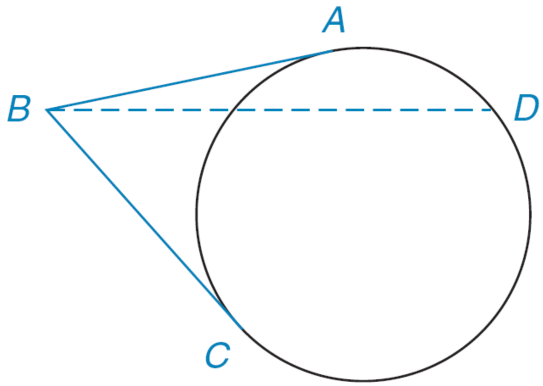


Figure 6.34(a)

More Angle Measures in the Circle

Theorem 6.2.7

If an angle is formed by two intersecting tangents, then the measure of the angle is one-half the difference of the measures of the intercepted arcs.

In Figure 6.34(a), $\angle ABC$ intercepts the two arcs determined by points *A* and *C*. The small arc is a minor arc $(\overset{\frown}{AC})$, and the large arc is a major arc $(\overset{\frown}{ADC})$.

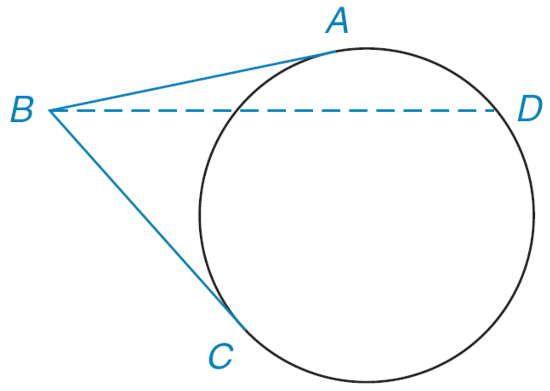


Figure 6.34(a)

More Angle Measures in the Circle

According to Theorem 6.2.7,

$$\mathrm{m}\angle ABC = \frac{1}{2}(\mathrm{m}\overset{\frown}{ADC} - \mathrm{m}\overset{\frown}{AC}).$$

More Angle Measures in the Circle

Let’s review the methods used to measure the different types of angles related to a circle. These are summarized in Table 6.1.

TABLE 6.1

Methods for Measuring Angles Related to a Circle

Location of the Vertex of the Angle	**Rule for Measuring the Angle**
Center of the circle (central angle)	The *measure* of the intercepted arc
In the *interior* of the circle (interior angle)	*One-half the sum* of the measures of the intercepted arcs
On the circle (inscribed angle)	*One-half the measure* of the intercepted arc
In the *exterior* of the circle (exterior angle)	*One-half the difference* of the measures of the two intercepted arcs

More Angle Measures in the Circle

Theorem 6.2.8

If two parallel lines intersect a circle, the intercepted arcs between these lines are congruent.

Where $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ in Figure 6.36, it follows that $\overset{\frown}{AC} \cong \overset{\frown}{BD}$.
Equivalently, $m\overset{\frown}{AC} = m\overset{\frown}{BD}$.

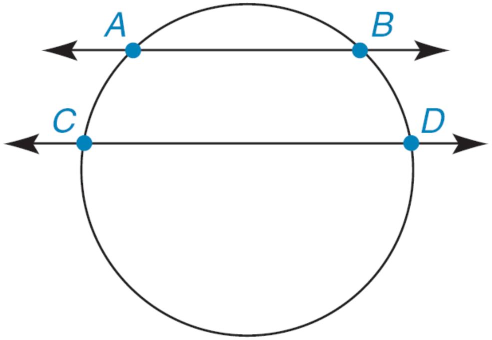


Figure 6.36