Chapter 6

Circles

6.1 Circles and Related Segments and Angles

Circles and Related Segments and Angles

Definition

A **circle** is the set of all points in a plane that are at a fixed distance from a given point known as the *center* of the circle.

Circles and Related Segments and Angles

A circle is named by its center point. In Figure 6.1, point *P* is the center of the circle.

The symbol for circle is ⊙, so the circle in Figure 6.1 is ⊙ *P*.

Points *A*, *B*, *C*, and *D* are points *of* (or *on*) the circle.

Points *P* (the center) and *R* are in the *interior* of circle *P*; points *G* and *H* are in the *exterior* of the circle.

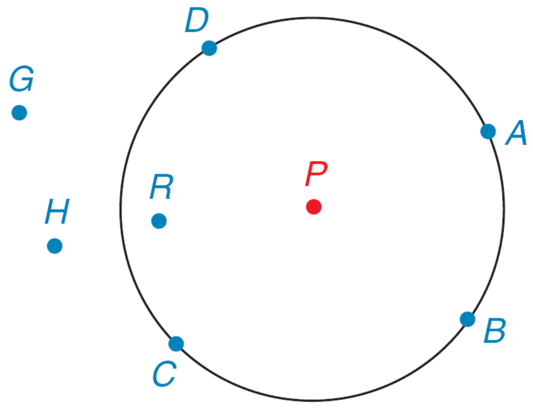


Figure 6.1

Circles and Related Segments and Angles

In $\odot Q$ of Figure 6.2, $\overline{SQ}$ is a radius of the circle. A **radius** is a segment that joins the center of the circle to a point on the circle.

$\overline{SQ}$, $\overline{TQ}$, $\overline{VQ}$, and $\overline{WQ}$ are **radii** (plural of *radius*) of $\odot Q$. By definition, $SQ = TQ = VQ = WQ$.

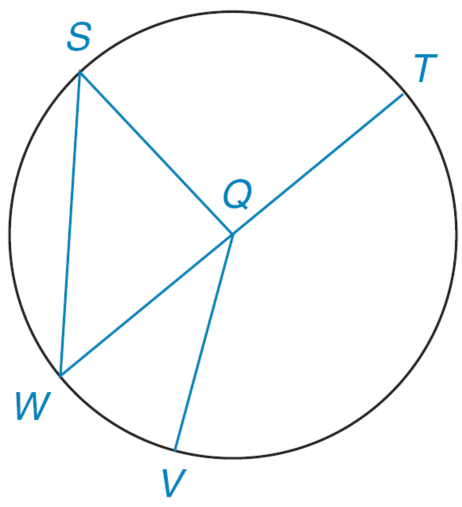


Figure 6.2

Circles and Related Segments and Angles

The following statement is a consequence of the definition of a circle.

All radii of a circle are congruent.

A line segment (such as $\overline{SW}$ in Figure 6.2) that joins two points of a circle is a **chord** of the circle.

A **diameter** of a circle is a chord that contains the center of the circle; if *T*-*Q*-*W* in Figure 6.2, then $\overline{TW}$ is a diameter of $\odot Q$.

Circles and Related Segments and Angles

Definition

Congruent circles are two or more circles that have congruent radii.

In Figure 6.3, circles *P* and *Q* are congruent because their radii have equal lengths. We can slide $\odot P$ to the right to coincide with $\odot Q$.

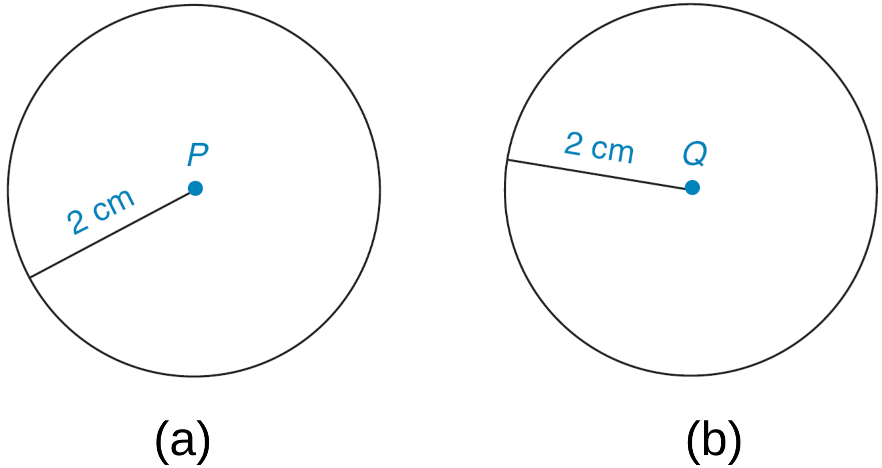


Figure 6.3

Definition

Concentric circles are coplanar circles that have a common center.

The concentric circles in Figure 6.4 have the common center *O*.

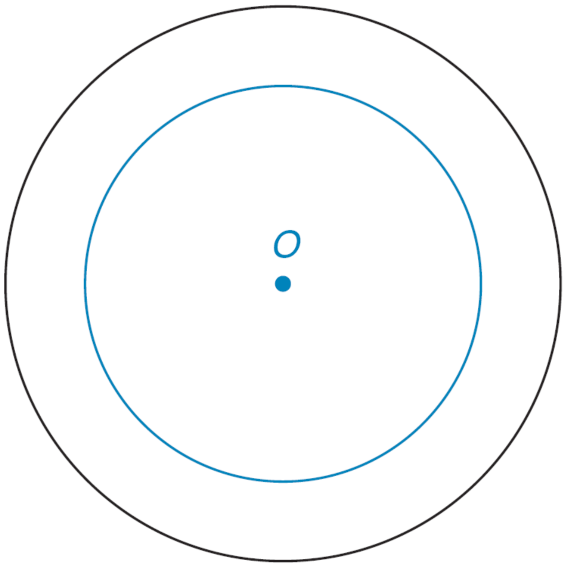


Figure 6.4

Circles and Related Segments and Angles

In $\odot P$ of Figure 6.5, the part of the circle shown in red from point A to point B is **arc** AB, symbolized by $\overset{\frown}{AB}$.

If $\overline{AC}$ is a diameter, then $\overset{\frown}{ABC}$ (three letters are used for clarity) is a **semicircle.**

In Figure 6.5, a **minor arc** like $\overset{\frown}{AB}$ is part of a semicircle; a **major arc** such as $\overset{\frown}{ABCD}$ (also denoted by $\overset{\frown}{ABD}$ or $\overset{\frown}{ACD}$) is more than a semicircle but less than the entire circle.

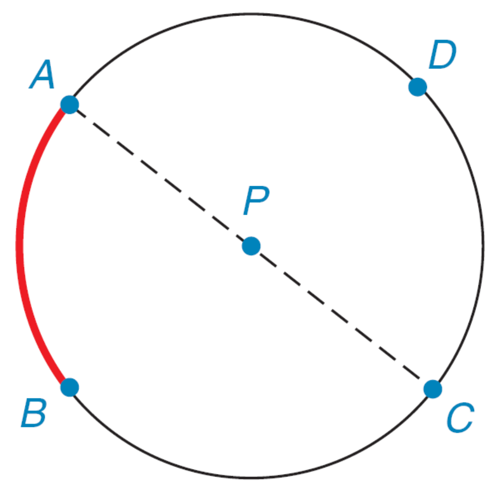


Figure 6.5

Circles and Related Segments and Angles

Definition

A **central angle** of a circle is an angle whose vertex is the center of the circle and whose sides are radii of the circle.

In Figure 6.6, $\angle NOP$ is a central angle of $\odot O$. The **intercepted arc** of $\angle NOP$ is $\overset{\frown}{NP}$.

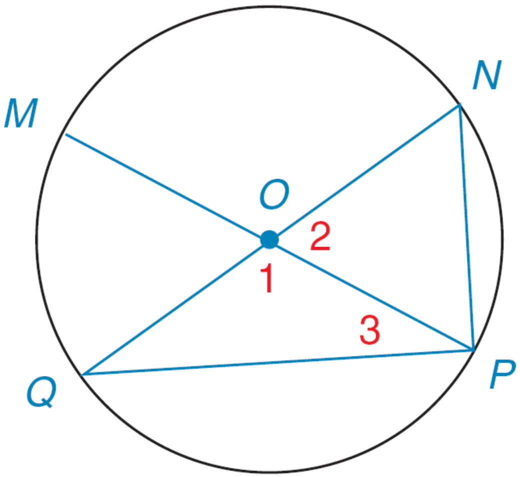


Figure 6.6

Circles and Related Segments and Angles

The intercepted arc of an angle is determined by the two points of intersection of the angle with the circle and all points of the arc in the interior of the angle.

Example 1

In Figure 6.6, $\overline{MP}$ and $\overline{NQ}$ intersect at *O*, the center of the circle. Name:

a) All four radii

b) Both diameters

c) All four chords

d) One central angle

e) One minor arc

f) One semicircle

g) One major arc

h) Intercepted arc of $\angle MON$

i) Central angle that intercepts $\overset{\frown}{NP}$

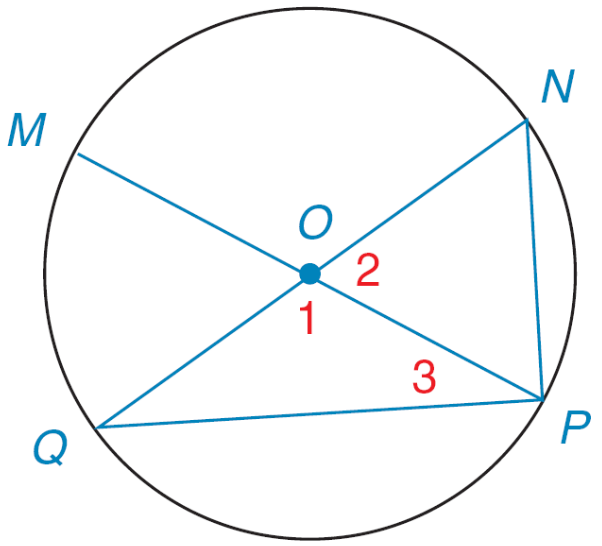


Figure 6.6

Example 1 – *Solution*

a) $\overline{OM}$, $\overline{OQ}$, $\overline{OP}$, and $\overline{ON}$

b) $\overline{MP}$ and $\overline{QN}$

c) $\overline{MP}$, $\overline{QN}$, $\overline{QP}$, and $\overline{NP}$

d) $\angle QOP$ (other answers are possible)

e) $\overset{\frown}{NP}$ (other answers are possible)

Example 1 – *Solution*

cont'd

f) $\overset{\frown}{MQP}$ (other answers are possible)

g) $\overset{\frown}{MQN}$ (can be named $\overset{\frown}{MQPN}$; other answers are possible)

h) $\overset{\frown}{MN}$ (lies in the interior of $\angle$ *MON* and includes points *M* and *N*)

i) $\angle$ *NOP* (also called $\angle$ 2)

Circles and Related Segments and Angles

The following statement is a consequence of the Segment-Addition Postulate.

In a circle, the length of a diameter is twice the length of a radius; in symbols, $d = 2r$.

Theorem 6.1.1

A radius that is perpendicular to a chord bisects the chord.

ANGLE AND ARC RELATIONSHIPS IN THE CIRCLE

Angle and Arc Relationships in the Circle

In Figure 6.8, the sum of the measures of the angles about point *O* (angles determined by perpendicular diameters $\overline{AC}$ and $\overline{BD}$) is 360°.

Similarly, the circle can be separated into 360 equal arcs, *each of which measures 1° of arc measure;* that is, each arc would be intercepted by a central angle measuring 1°.

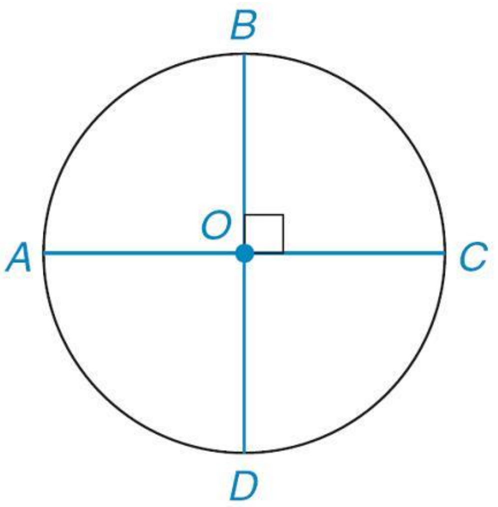


Figure 6.8

Angle and Arc Relationships in the Circle

Our description of arc measure leads to the following postulate.

Postulate 16 (Central Angle Postulate)

In a circle, the degree measure of a central angle is equal to the degree measure of its intercepted arc.

If $m\overset{\frown}{AB} = 90°$ in Figure 6.8, then $m\angle AOB = 90°$.

The reflex angle that intercepts $\overset{\frown}{BCA}$ and that is composed of three right angles measures 270°.

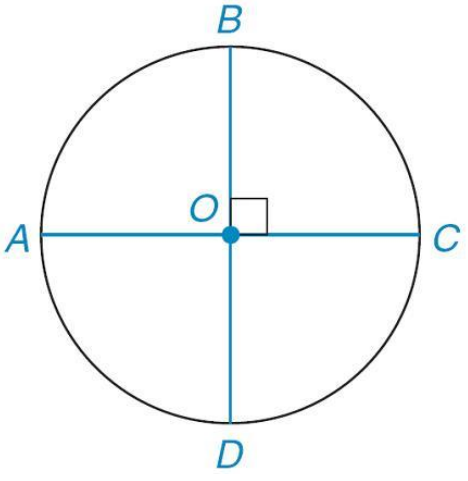


Figure 6.8

Angle and Arc Relationships in the Circle

In figure 6.8, m$\overset{\frown}{AB}$ = 90°, m$\overset{\frown}{BCD}$ = 180°, and m$\overset{\frown}{AD}$ = 90°.

It follows that m$\overset{\frown}{AB}$ + m$\overset{\frown}{BCD}$ + m$\overset{\frown}{AD}$ = 360°.

Angle and Arc Relationships in the Circle

Consequently, we have the following generalization.

The sum of the measures of the consecutive arcs that form a circle is 360°.

In $\odot Y$ [Figure 6.9(a)], if $m\angle XYZ = 76°$, then $m\overset{\frown}{XZ} = 76°$ by the Central Angle Postulate.

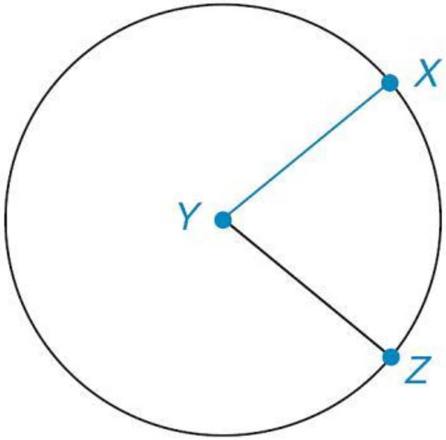
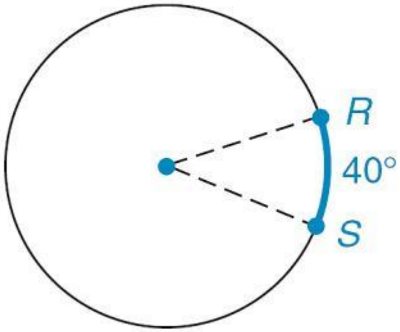


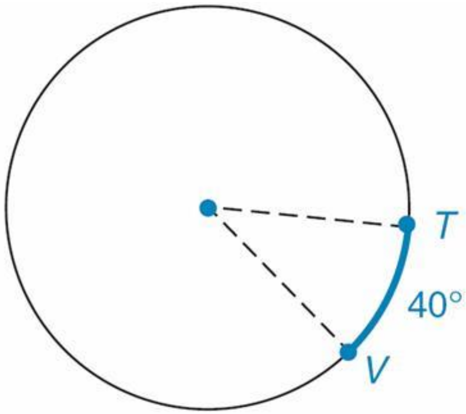
Figure 6.9(a)

Angle and Arc Relationships in the Circle

If two arcs have equal degree measures [Figures 6.9(b) and (c)] but are parts of two circles with unequal radii, then these arcs will not coincide.



(b)



(c)

Figure 6.9

Angle and Arc Relationships in the Circle

This observation leads to the following definition.

Definition

In a circle or congruent circles, **congruent arcs** are arcs with equal measures.

Angle and Arc Relationships in the Circle

To further clarify the definition of congruent arcs, consider the concentric circles (having the same center) in Figure 6.10.

Here the degree measure of $\angle AOB$ of the smaller circle is the same as the degree measure of $\angle COD$ of the larger circle.

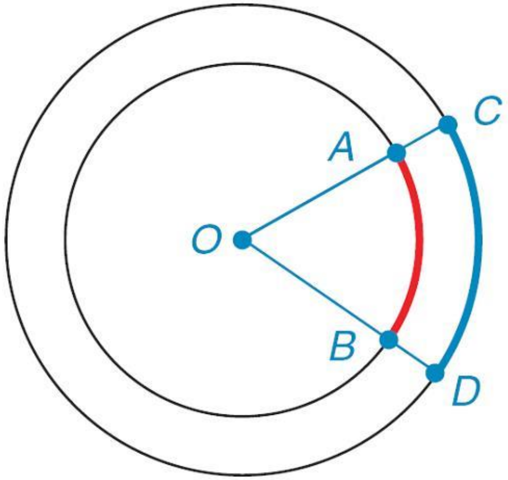


Figure 6.10

Even though $m\overset{\frown}{AB} = m\overset{\frown}{CD}$,

we conclude that $\overset{\frown}{AB} \ncong \overset{\frown}{CD}$

because the arcs would not coincide.

Example 3

In $\odot O$ of Figure 6.11, $\overrightarrow{OE}$ bisects $\angle AOD$.

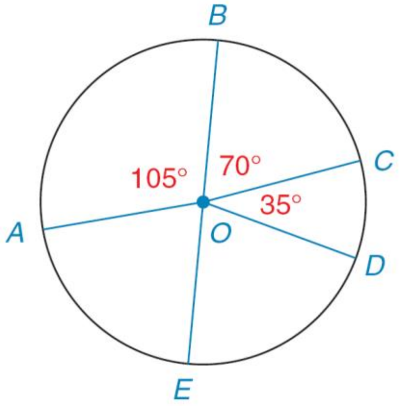


Figure 6.11

Using the measures indicated, find:

a) $m\overset{\frown}{AB}$ **b)** $m\overset{\frown}{BC}$ **c)** $m\overset{\frown}{BD}$ **d)** $m\angle AOD$

e) $m\overset{\frown}{AE}$ **f)** $m\overset{\frown}{ACE}$ **g)** whether $\overset{\frown}{AE} \cong \overset{\frown}{ED}$

h) The measure of the reflex angle that intercepts $\overset{\frown}{ABCD}$

Example 3 – *Solution*

a) 105°

b) 70°

c) 105°

d) 150°, from 360 – (105 + 70 + 35)

e) 75° because the corresponding central angle ($\angle AOE$) is the result of bisecting $\angle AOD$, which was found to be 150°

Example 3 – *Solution*

cont'd

f) 285° (from 360 – 75, the measure of $\overset{\frown}{AE}$)

g) The arcs are congruent because both measure 75° and are arcs in the same circle.

h) 210° (from 105° + 70° + 35°)

Angle and Arc Relationships in the Circle

In Figure 6.11, note that $m\overparen{BC} + m\overparen{CD} = m\overparen{BD}$ (or $m\overparen{BCD}$).

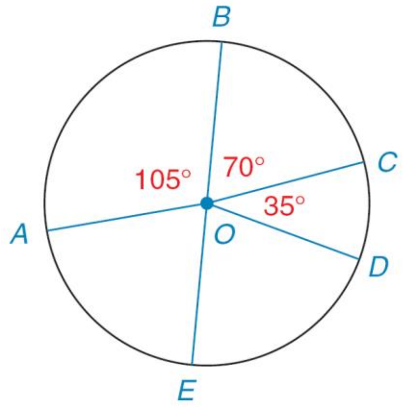


Figure 6.11

Because the union of $\overparen{BD}$ and $\overparen{DA}$ is the major arc $\overparen{BDA}$, we also see that $m\overparen{BD} + m\overparen{DA} = m\overparen{BDA}$.

Angle and Arc Relationships in the Circle

With the understanding that $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$ do not overlap, we generalize the relationship as follows.

Postulate 17 (Arc-Addition Postulate)

If $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$ intersect only at point *B*, then

$$m\overset{\frown}{AB} + m\overset{\frown}{BC} = m\overset{\frown}{ABC}.$$

Given points *A*, *B*, and *C* on $\odot$ *O* as shown in Figure 6.12(a), suppose that radii $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$ are drawn.

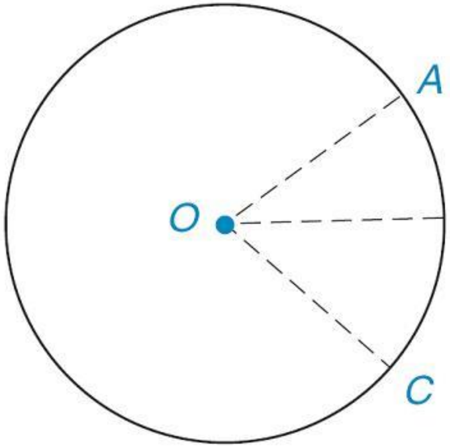


Figure 6.12(a)

Angle and Arc Relationships in the Circle

Just as $m\angle AOB + m\angle BOC = m\angle AOC$

by the Angle-Addition Postulate, it follows that

$$m\overset{\frown}{AB} + m\overset{\frown}{BC} = m\overset{\frown}{ABC}$$

In the statement of the Arc-Addition Postulate, the reason for writing $\overset{\frown}{ABC}$ (rather than $\overset{\frown}{AC}$) is that the arc with endpoints at *A* and *C could be* a major arc.

The Arc-Addition Postulate can easily be extended to include more than two arcs.

Angle and Arc Relationships in the Circle

In Figure 6.12(b), $\text{m}\,\overset{\frown}{RS} + \text{m}\,\overset{\frown}{ST} + \text{m}\,\overset{\frown}{TQ} = \text{m}\,\overset{\frown}{RSTQ}$.

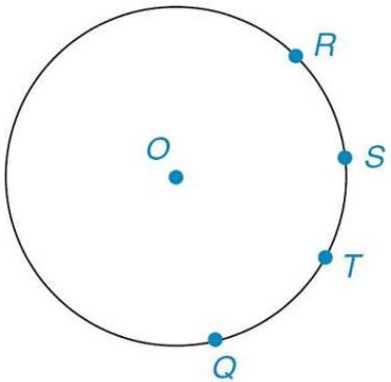


Figure 6.12(b)

If $\text{m}\,\overset{\frown}{RS} = \text{m}\,\overset{\frown}{ST}$, then point *S* is the **midpoint** of $\overset{\frown}{RT}$; alternately, $\overset{\frown}{RT}$ is **bisected** at point *S*.

Angle and Arc Relationships in the Circle

Definition

An **inscribed angle** of a circle is an angle whose vertex is a point on the circle and whose sides are chords of the circle.

Theorem 6.1.2

The measure of an inscribed angle of a circle is one-half the measure of its intercepted arc.

Angle and Arc Relationships in the Circle

Theorem 6.1.3

In a circle (or in congruent circles), congruent minor arcs have congruent central angles.

Theorem 6.1.4

In a circle (or in congruent circles), congruent central angles have congruent arcs.

Theorem 6.1.5

In a circle (or in congruent circles), congruent chords have congruent minor (major) arcs.

Angle and Arc Relationships in the Circle

Theorem 6.1.6

In a circle (or in congruent circles), congruent arcs have congruent chords.

Theorem 6.1.7

Chords that are at the same distance from the center of a circle are congruent.

Angle and Arc Relationships in the Circle

Theorem 6.1.8

Congruent chords are located at the same distance from the center of a circle.

Theorem 6.1.9

An angle inscribed in a semicircle is a right angle.

Theorem 6.1.10

If two inscribed angles intercept the same arc, then these angles are congruent.