Chapter 5

Similar Triangles

5.6 Segments Divided Proportionally

Segments Divided Proportionally

In this section, we begin with an informal description of the phrase *divided proportionally*.

Suppose that three children have been provided with a joint savings account by their parents.

Equal monthly deposits have been made to the account for each child since birth.

Segments Divided Proportionally

If the ages of the children are 2, 4, and 6 (assume exactness of ages for simplicity) and the total in the account is $7200, then the amount that each child should receive can be found by solving the equation

$$2x + 4x + 6x = 7200$$

Solving this equation leads to the solution $1200 for the 2-year-old, $2400 for the 4-year-old, and $3600 for the 6-year-old.

Segments Divided Proportionally

We say that the amount has been divided proportionally.

Expressed as a proportion that compares the amount saved to age, we have

$$\frac{1200}{2} = \frac{2400}{4} = \frac{3600}{6}$$

Segments Divided Proportionally

In Figure 5.44, $\overline{AC}$ and $\overline{DF}$ are divided proportionally at points *B* and *E* if

$$\frac{AB}{DE} = \frac{BC}{EF} \quad \text{or} \quad \frac{AB}{BC} = \frac{DE}{EF}$$

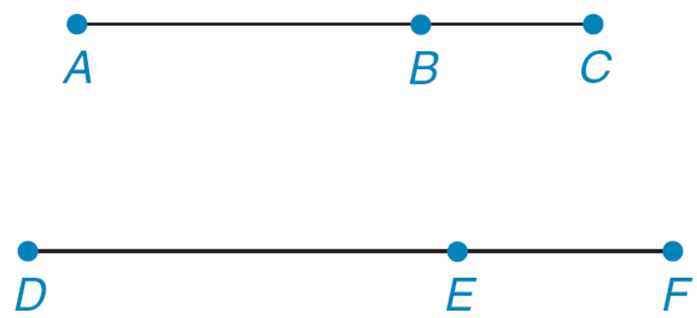

Figure 5.44

Segments Divided Proportionally

Of course, a pair of segments may be divided proportionally by several points. In Figure 5.45, $\overline{RW}$ and $\overline{HM}$ are divided proportionally when

$$\frac{RS}{HJ} = \frac{ST}{JK} = \frac{TV}{KL} = \frac{VW}{LM} \qquad \left(\textbf{notice that } \frac{6}{4} = \frac{12}{8} = \frac{15}{10} = \frac{9}{6}\right)$$

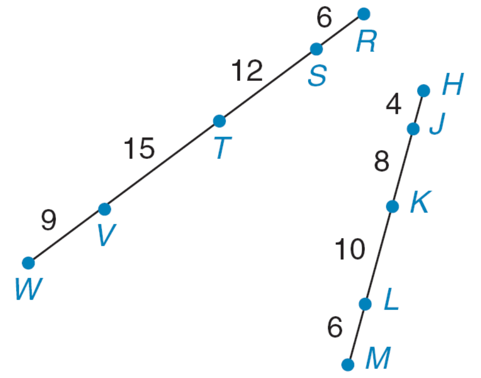

Figure 5.45

Example 1

In Figure 5.46, points *D* and *E* divide $\overline{AB}$ and $\overline{AC}$ proportionally. If *AD* = 4, *DB* = 7, and *EC* = 6, find *AE*.

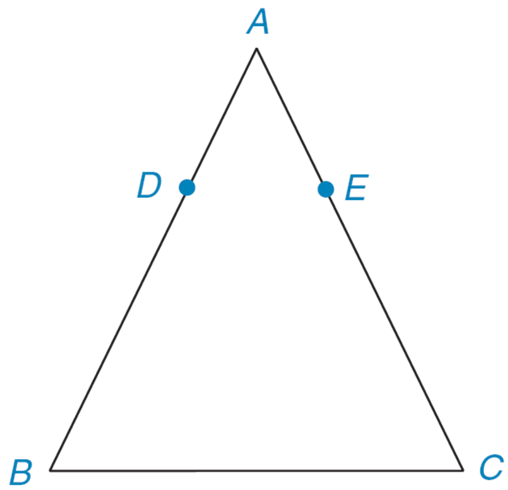

Figure 5.46

Example 1 – *Solution*

$$\frac{AD}{AE} = \frac{DB}{EC}.$$

Where $AE = x$, $\frac{4}{x} = \frac{7}{6}.$

Then $7x = 24$,

so $x = AE = \frac{24}{7}$

$= 3\frac{3}{7}.$

Segments Divided Proportionally

NUMERATOR-DENOMINATOR ADDITION PROPERTY

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a + c}{b + d} = \frac{a}{b} = \frac{c}{d}$$

In words, we may restate this property as follows:

The fraction whose numerator and denominator are determined, respectively, by adding numerators and denominators of equal fractions is equal to each of those equal fractions.

Here is a numerical example of this claim:

$$\text{If } \frac{2}{3} = \frac{4}{6}, \text{ then } \frac{2 + 4}{3 + 6} = \frac{2}{3} = \frac{4}{6}$$

Segments Divided Proportionally

Two properties that were introduced are now recalled.

If $\frac{a}{b} = \frac{c}{d}$, , then $\frac{a \pm b}{b} = \frac{c \pm d}{d}$

Theorem 5.6.1

If a line is parallel to one side of a triangle and intersects the other two sides, then it divides these sides proportionally.

Segments Divided Proportionally

Corollary 5.6.2

When three (or more) parallel lines are cut by a pair of transversals, the transversals are divided proportionally by the parallel lines.

Theorem 5.6.3 (The Angle-Bisector Theorem)

If a ray bisects one angle of a triangle, then it divides the opposite side into segments whose lengths are proportional to the lengths of the two sides that form the bisected angle.

Segments Divided Proportionally

The “Prove statement” of the preceding theorem indicates that one form of the proportion described is given by comparing lengths as shown:

$$\frac{\text{segment at left}}{\text{side at left}} = \frac{\text{segment at right}}{\text{side at right}}$$

Equivalently, the proportion could compare lengths like this:

$$\frac{\text{segment at left}}{\text{segment at right}} = \frac{\text{side at left}}{\text{side at right}}$$

Other forms of the proportion are also possible!

CEVA'S THEOREM

Ceva's Theorem

Theorem 5.6.4 (Ceva's Theorem)

Let point D be any point in the interior of $\triangle ABC$. Where E, F, and G lie on $\triangle ABC$, let $\overline{BE}$, $\overline{AF}$, and $\overline{CG}$ be the line segments determined by D and vertices of $\triangle ABC$.

Then the product of the ratios of the lengths of the segments of each of the three sides (taken in order from a given vertex of the triangle) equals 1; that is,

$$\frac{AG}{GB} \cdot \frac{BF}{FC} \cdot \frac{CE}{EA} = 1$$

Ceva's Theorem

For Figure 5.53, Ceva's Theorem can be stated in many equivalent forms:

$$\frac{AE}{EC} \cdot \frac{CF}{FB} \cdot \frac{BG}{GA} = 1, \quad \frac{CF}{FB} \cdot \frac{BG}{GA} \cdot \frac{AE}{EC} = 1, \text{ etc.}$$

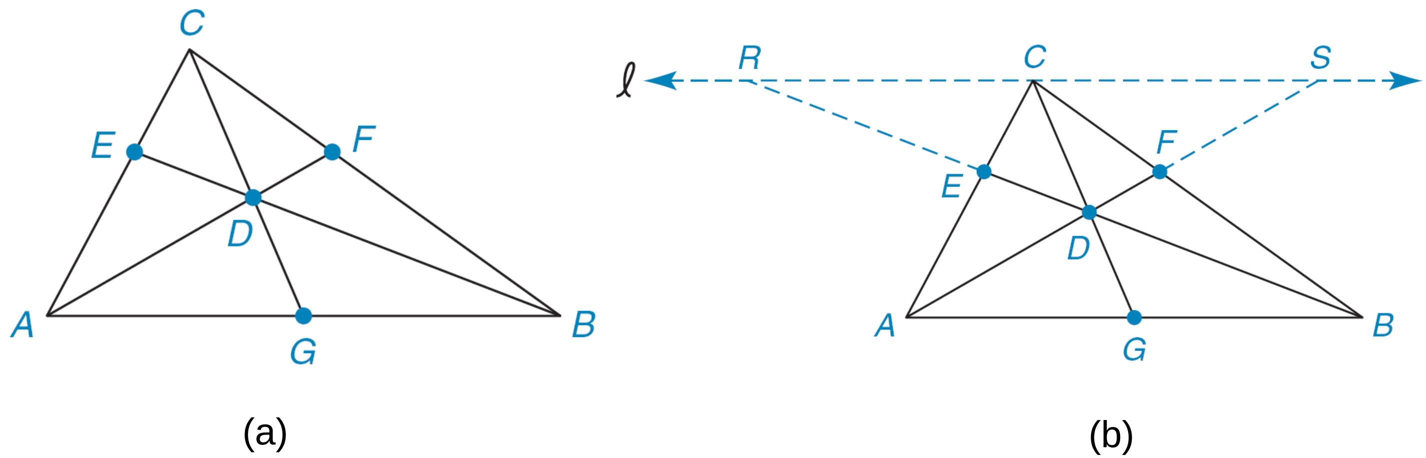

Figure 5.53

Ceva’s Theorem

In each case, we select a vertex (such as A) and form the product of the ratios of the lengths of segments of sides in a set order (clockwise or counterclockwise).