Chapter 5 Similar Triangles

5.4 The Pythagorean Theorem

The Pythagorean Theorem

The following theorem will enable us to prove the well-known Pythagorean Theorem.

Theorem 5.4.1

The altitude drawn to the hypotenuse of a right triangle separates the right triangle into two right triangles that are similar to each other and to the original right triangle.

The Pythagorean Theorem

Theorem 5.4.1 is illustrated by Figure 5.18, in which the right triangle $\triangle ABC$ has its right angle at vertex C so that $\overline{CD}$ is the altitude to hypotenuse $\overline{AB}$.

The smaller triangles are shown in Figures 5.18(b) and (c), and the original triangle is shown in Figure 5.18(d).

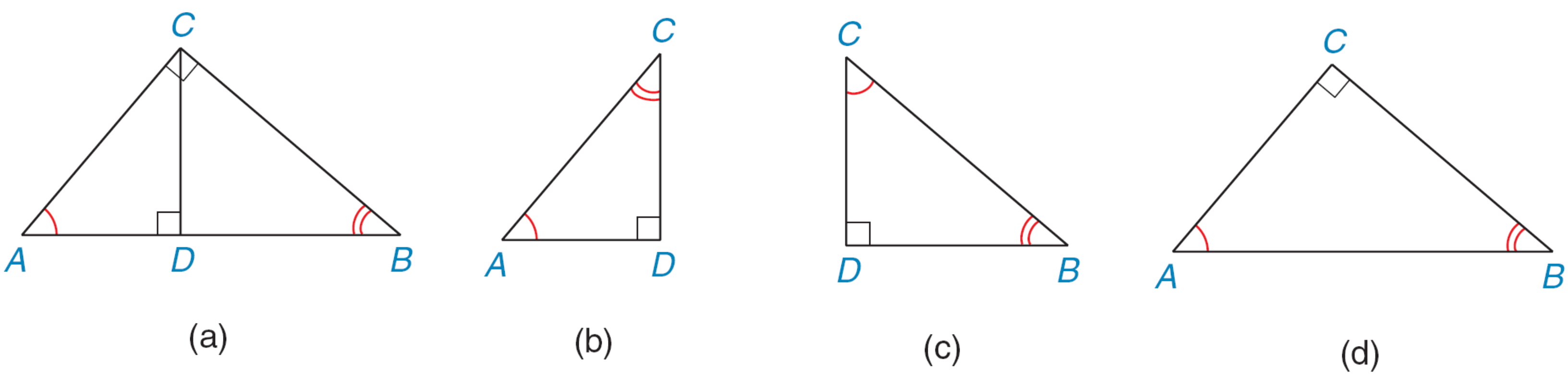

Figure 5.18

Note the matching arcs indicating congruent angles.

The Pythagorean Theorem

In Figure 5.18(a), $\overline{AD}$ and $\overline{DB}$ are known as *segments* (parts) of the hypotenuse $\overline{AB}$.

Furthermore, $\overline{AD}$ is the segment of the hypotenuse *adjacent* to (next to) leg $\overline{AC}$, and $\overline{BD}$ is the segment of the hypotenuse *adjacent* to leg $\overline{BC}$.

The Pythagorean Theorem

Theorem 5.4.2

The length of the altitude to the hypotenuse of a right triangle is the geometric mean of the lengths of the segments of the hypotenuse.

Lemma 5.4.3

The length of each leg of a right triangle is the geometric mean of the length of the hypotenuse and the length of the segment of the hypotenuse adjacent to that leg.

The Pythagorean Theorem

Theorem 5.4.4 (Pythagorean Theorem)

The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the legs.

Example 1

Given $\triangle RST$ with right $\angle S$ in Figure 5.23, find:

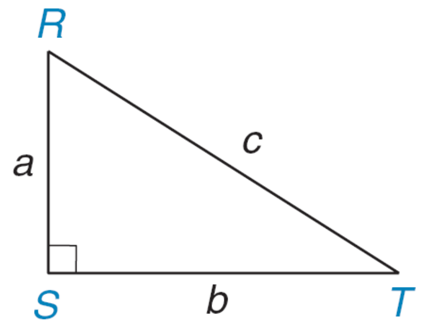

Figure 5.23

a) RT if $RS = 3$ and $ST = 4$

b) ST if $RS = 6$ and $RT = 9$

Example 1 – *Solution*

With right $\angle S$, the hypotenuse is $\overline{RT}$. Then $RT = c$, $RS = a$, and $ST = b$.

a) $3^2 + 4^2 = c^2 \rightarrow 9 + 16 = c^2$

$$c^2 = 25$$

$$c = 5;$$

$$RT = 5$$

Example 1 – *Solution*

cont'd

b) $6^2 + b^2 = 9^2 \rightarrow 36 + b^2 = 81$

$$b^2 = 45$$

$$b = \sqrt{45}$$

$$= \sqrt{9 \cdot 5}$$

$$= \sqrt{9} \cdot \sqrt{5}$$

$$= 3\sqrt{5}$$

$$ST = 3\sqrt{5}$$

$$\approx 6.71$$

The Pythagorean Theorem

Theorem 5.4.5 (Converse of Pythagorean Theorem)

If *a*, *b*, and *c* are the lengths of the three sides of a triangle, with *c* the length of the longest side, and if $c^2 = a^2 + b^2$, then the triangle is a right triangle with the right angle opposite the side of length *c*.

Theorem 5.4.6

If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and a leg of a second right triangle, then the triangles are congruent (HL).

The Pythagorean Theorem

Our work with the Pythagorean Theorem would be incomplete if we did not address two issues.

The first, Pythagorean triples, involves natural (or counting) numbers as possible choices of a, b, and c.

The second leads to the classification of triangles according to the lengths of their sides.

PYTHAGOREAN TRIPLES

Pythagorean Triples

Definition

A **Pythagorean triple** is a set of three natural numbers (a, b, c) for which $a^2 + b^2 = c^2$.

Three sets of Pythagorean triples encountered in this section are (3, 4, 5), (5, 12, 13),and (8, 15, 17).

These combinations of numbers, as lengths of sides, always lead to a right triangle.

Pythagorean Triples

Natural-number multiples of any of these triples also produce Pythagorean triples.

For example, doubling (3, 4, 5) yields (6, 8, 10), which is also a Pythagorean triple. In Figure 5.29, the triangles are similar by SSS~.

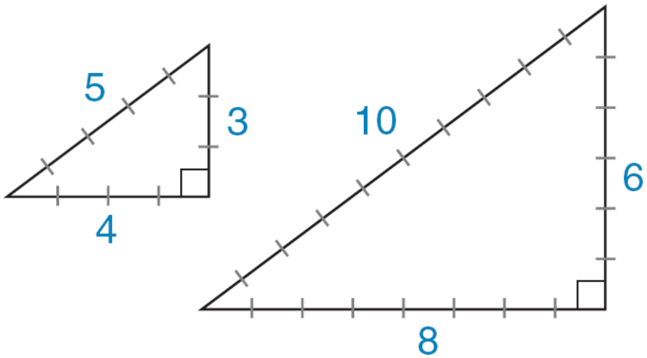

Figure 5.29

Pythagorean Triples

The Pythagorean triple (3, 4, 5) also leads to the multiples (9, 12, 15), (12, 16, 20), and (15, 20, 25). The Pythagorean triple (5, 12, 13) leads to triples such as (10, 24, 26) and (15, 36, 39).

Basic Pythagorean triples that are used less frequently include (7, 24, 25), (9, 40, 41), and (20, 21, 29).

Pythagorean Triples

Pythagorean triples can be generated by using select formulas.

Where p and q are natural numbers and $p > q$, one formula uses $2pq$ for the length of one leg, $p^2 - q^2$ for the length of other leg, and $p^2 + q^2$ for the length of the hypotenuse (See Figure 5.30.).

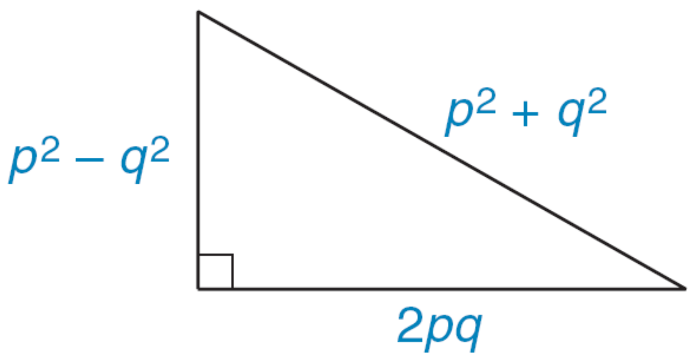

Figure 5.30

Pythagorean Triples

Table 5.1 lists some Pythagorean triples corresponding to choices for *p* and *q*.

TABLE 5.1

Pythagorean Triples

p	q	a (or b) $p^2 - q^2$	b (or a) $2pq$	c $p^2 + q^2$	(a, b, c)
2	1	3	4	5	**(3, 4, 5)**
3	1	8	6	10	(6, 8, 10)
3	2	5	12	13	**(5, 12, 13)**
4	1	15	8	17	**(8, 15, 17)**
4	3	7	24	25	**(7, 24, 25)**
5	1	24	10	26	(10, 24, 26)
5	2	21	20	29	**(20, 21, 29)**
5	3	16	30	34	(16, 30, 34)
5	4	9	40	41	**(9, 40, 41)**

Pythagorean Triples

The triples printed in boldface type are *basic triples*, also known as *primitive triples*.

In application, knowledge of the primitive triples and their multiples will save you considerable time and effort.

In the final column, the resulting triple is provided in the order from *a* (small) to *c* (large).

THE CONVERSE OF THE PYTHAGOREAN THEOREM

The Converse of the Pythagorean Theorem

The Converse of the Pythagorean Theorem allows us to recognize a right triangle by knowing the lengths of its sides.

A variation on this converse allows us to determine whether a triangle is acute or obtuse.

The Converse of the Pythagorean Theorem

Theorem 5.4.7

Let a, b, and c represent the lengths of the three sides of a triangle, with c the length of the longest side.

1. If $c^2 > a^2 + b^2$, then the triangle is obtuse and the obtuse angle lies opposite the side of length c.

2. If $c^2 < a^2 + b^2$, then the triangle is acute.

Example 6

Determine the type of triangle represented if the lengths of its sides are as follows:

a) 4, 5, 7

b) 6, 7, 8

c) 9, 12, 15

d) 3, 4, 9

Example 6 – *Solution*

a) Choosing $c = 7$, we have $7^2 > 4^2 + 5^2$, so the triangle is obtuse.

b) Choosing $c = 8$, we have $8^2 < 6^2 + 7^2$, so the triangle is acute.

c) Choosing $c = 15$, we have $15^2 = 9^2 + 12^2$, so the triangle is a right triangle.

d) Because $9 > 3 + 4$, no triangle is possible. (Remember that the sum of the lengths of two sides of a triangle must be greater than the length of the third side.)