Chapter 5

Similar Triangles

5.1 Ratios, Rates, and Proportions

Ratios, Rates, and Proportions

A **ratio** is the quotient $\frac{a}{b}$ (where $b \neq 0$) that provides a comparison between the numbers *a* and *b*.

Because every fraction indicates a division, every fraction represents a ratio. Read "*a* to *b*," the ratio is sometimes written in the form *a*:*b*. The numbers *a* and *b* are often called the *terms* of the ratio.

It is generally preferable to express the ratio in simplified form (lowest terms), so the ratio 6 to 8 would be reduced (in fraction form) from $\frac{6}{8}$ to $\frac{3}{4}$.

Ratios, Rates, and Proportions

If units of measure are found in a ratio, these units must be **commensurable** (convertible to the same unit of measure).

When simplifying the ratio of two quantities that are expressed in the same unit, we eliminate the common unit in the process.

If two quantities cannot be compared because no common unit of measure is possible, the quantities are said to be **incommensurable.**

Example 1

Find the best form of each ratio:

a) 12 to 20

b) 12 in. to 24 in.

c) 12 in. to 3 ft (**Note:** 1 ft = 12 in.)

d) 5 lb to 20 oz (**Note:** 1 lb = 16 oz)

e) 5 lb to 2 ft

f) 4 m to 30 cm (**Note:** 1 m = 100 cm)

Example 1 – *Solution*

a) $\dfrac{12}{20} = \dfrac{3}{5}$

b) $\dfrac{12\ \text{in.}}{24\ \text{in.}} = \dfrac{12}{24} = \dfrac{1}{2}$

c) $\dfrac{12\,\text{in.}}{3\,\text{ft}} = \dfrac{12\,\text{in.}}{3(12\,\text{in.})} = \dfrac{12\,\text{in.}}{36\,\text{in.}} = \dfrac{1}{3}$

d) $\dfrac{5\,\text{lb}}{20\ \text{oz}} = \dfrac{5(16\ \text{oz})}{20\ \text{oz}} = \dfrac{80\ \text{oz}}{20\ \text{oz}} = \dfrac{4}{1}$

Example 1 – *Solution*

cont'd

e) $\dfrac{5\,\text{lb}}{2\,\text{ft}}$ is incommensurable!

f) $$\frac{4\,\text{m}}{30\,\text{cm}} = \frac{4(100\,\text{cm})}{30\,\text{cm}} = \frac{400\,\text{cm}}{30\,\text{cm}} = \frac{40}{3}$$

Ratios, Rates, and Proportions

A **rate** is a quotient that relates two quantities that are incommensurable.

If an automobile can travel 300 miles along an interstate while consuming 10 gallons of gasoline, then its consumption *rate* is $\frac{300\text{ miles}}{10\text{ gallons}}$.

In simplified form, the consumption rate is $\frac{30\text{ mi}}{\text{gal}}$, which is read as “30 miles per gallon” and abbreviated 30 mpg.

Ratios, Rates, and Proportions

A **proportion** is a statement that equates two ratios or two rates.

Thus, $\frac{a}{b} = \frac{c}{d}$ is a proportion and may be read as "*a* is to *b* as *c* is to *d.*" In the order read, *a* is the *first term* of the proportion, *b* is the *second term, c* is the *third term,* and *d* is the *fourth term.*

The first and last terms (*a* and *d*) of the proportion are the **extremes,** whereas the second and third terms (*b* and *c*) are the **means.**

Ratios, Rates, and Proportions

The following property is extremely convenient for solving proportions.

Property 1 (Means-Extremes Property)

In a proportion, the product of the means equals the product of the extremes; that is, if $\frac{a}{b} = \frac{c}{d}$ (where $b \neq 0$ and $d \neq 0$), then $a \cdot d = b \cdot c$.

Ratios, Rates, and Proportions

Because a proportion is a statement, it could be true or false. In the false proportion $\frac{9}{12} = \frac{2}{3}$, it is obvious that $9 \cdot 3 \neq 12 \cdot 2$; on the other hand, the truth of the statement $\frac{9}{12} = \frac{3}{4}$ is evident from the fact that $9 \cdot 4 = 12 \cdot 3$.

Henceforth, any proportion given in this text is intended to be a true proportion.

In application problems involving proportions, it is essential to order the related quantities in each ratio or rate.

Ratios, Rates, and Proportions

Definition

The nonzero number *b* is the **geometric mean** of *a* and *c* if

$$\frac{a}{b} = \frac{b}{c} \text{ or } \frac{c}{b} = \frac{b}{a}.$$

For example, 6 and –6 are the geometric means of 4 and 9 because $\frac{4}{6} = \frac{6}{9}$ and $\frac{4}{-6} = \frac{-6}{9}$.

Because applications in geometry generally require positive solutions, we usually seek only the positive geometric mean of *a* and *c.*

Ratios, Rates, and Proportions

An **extended ratio** compares more than two quantities and must be expressed in a form such as *a:b:c* or *d:e:f:g.*

If you know that the angles of a triangle are 90°, 60°, and 30°, then the ratio that compares these measures is 90:60:30, or 3:2:1 (because 90, 60, and 30 have the greatest common factor of 30).

Property Of Extended Ratios

Unknown quantities in the ratio $a : b : c : d$ can be represented by ax, bx, cx, and dx.

Ratios, Rates, and Proportions

Property 2 (Alternative Forms of Proportions)

In a proportion, the means or the extremes (or both) may be interchanged; that is, if $\frac{a}{b} = \frac{c}{d}$ (where *a*, *b*, *c*, and *d* are nonzero), then $\frac{a}{c} = \frac{b}{d}$, $\frac{d}{b} = \frac{c}{a}$, and $\frac{d}{c} = \frac{b}{a}$.

Ratios, Rates, and Proportions

Given the proportion $\frac{2}{3} = \frac{8}{12}$, leads to conclusions such as

1. $\dfrac{2}{8} = \dfrac{3}{12}$ (means interchanged)

2. $\dfrac{12}{3} = \dfrac{8}{2}$ (extremes interchanged)

3. $\dfrac{3}{2} = \dfrac{12}{8}$ (both sides inverted; the result obtained when both means and extremes are interchanged)

Ratios, Rates, and Proportions

Property 3 (Sum and Difference Properties of a Proportion)

If $\frac{a}{b} = \frac{c}{d}$ (where $b \neq 0$ and $d \neq 0$), then

$$\frac{a+b}{b} = \frac{c+d}{d} \text{ and } \frac{a-b}{b} = \frac{c-d}{d}.$$

Given the proportion $\frac{2}{3} = \frac{8}{12}$, the Sum and Difference Property leads to conclusions such as

1. $\frac{2+3}{3} = \frac{8+12}{12}$ (each side simplifies to $\frac{5}{3}$)

2. $\frac{2-3}{3} = \frac{8-12}{12}$ (each side simplifies to $-\frac{1}{3}$)

Ratios, Rates, and Proportions

Just as there are extended ratios, there are also **extended proportions** such as

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \cdots$$

Suggested by different numbers of servings of a particular recipe, the statement below is an extended proportion comparing numbers of eggs to numbers of cups of milk:

$$\frac{2\text{ eggs}}{3\text{ cups}} = \frac{4\text{ eggs}}{6\text{ cups}} = \frac{6\text{ eggs}}{9\text{ cups}}$$