Chapter 3 Triangles

3.5 Inequalities in a Triangle

Inequalities in a Triangle

Important inequality relationships exist among the measured parts of a triangle.

To establish some of these, we recall and apply some facts from both algebra and geometry.

Inequalities in a Triangle

Definition

Let *a* and *b* be real numbers.
a > **b** (read "a is greater than b") if and only if there is a positive number *p* for which *a* = *b* + *p*.

For instance, 9 > 4, because there is the positive number 5 for which 9 = 4 + 5. Because 5 + 2 = 7, we also know that 7 > 2 and 7 > 5.

In geometry, let *A*-*B*-*C* on $\overline{AC}$ so that *AB* + *BC* = *AC*; then *AC* > *AB*, because *BC* is a positive number.

LEMMAS (HELPING THEOREMS)

Lemmas (Helping Theorems)

Each of the five statements that follow is called a **lemma**.

Lemma 3.5.1

If B is between A and C on $\overline{AC}$, then $AC > AB$ and $AC > BC$. (The measure of a line segment is greater than the measure of any of its parts. See Figure 3.45.)

Figure 3.45

Lemmas (Helping Theorems)

Lemma 3.5.2

If $\overrightarrow{BD}$ separates $\angle ABC$ into two parts ($\angle 1$ and $\angle 2$), then $m\angle ABC > m\angle 1$ and $m\angle ABC > m\angle 2$.

(The measure of an angle is greater than the measure of any of its parts. See Figure 3.46.)

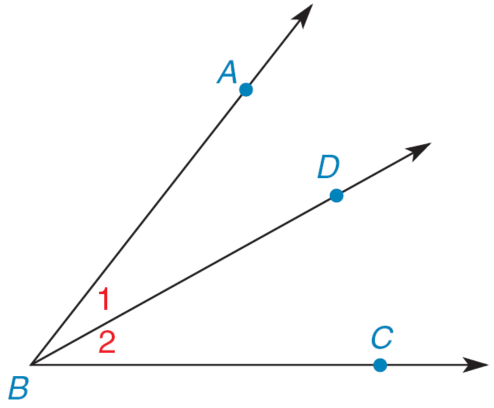

Figure 3.46

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Lemma 3.5.3

If $\angle 3$ is an exterior angle of a triangle and $\angle 1$ and $\angle 2$ are the nonadjacent interior angles, then $m\angle 3 > m\angle 1$ and $m\angle 3 > m\angle 2$. (The measure of an exterior angle of a triangle is greater than the measure of either nonadjacent interior angle. See Figure 3.47.)

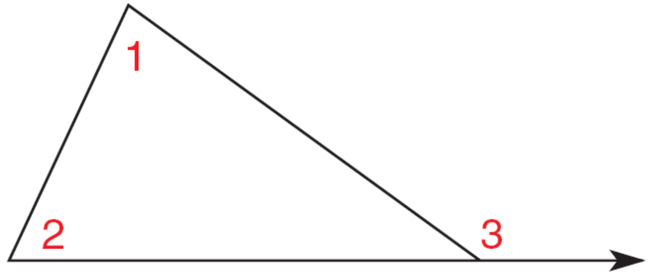

Figure 3.47

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Lemma 3.5.4

In $\triangle ABC$, if $\angle C$ is a right angle or an obtuse angle, then $m\angle C > m\angle A$ and $m\angle C > m\angle B$. (If a triangle contains a right or an obtuse angle, then the measure of this angle is greater than the measure of either of the remaining angles. See Figure 3.48.)

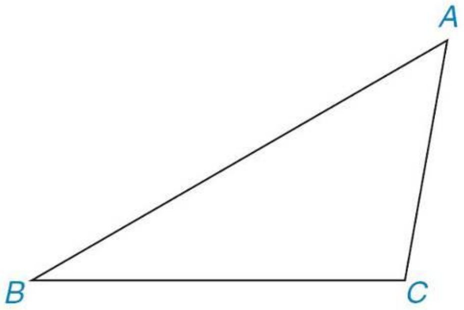

Figure 3.48

Lemma 3.5.5 (Addition Property of Inequality)

If $a > b$ and $c > d$, then $a + c > b + d$.

Example 1

Give a paragraph proof for the following problem.

Given: $AB > CD$ and $BC > DE$

Prove: $AC > CE$

Figure 3.49

Example 1

cont'd

Proof:

If $AB > CD$ and $BC > DE$, then $AB + BC > CD + DE$ by Lemma 3.5.5.

But $AB + BC = AC$ and $CD + DE = CE$ by the Segment-Addition Postulate.

Using substitution, it follows that $AC > CE$.

Lemmas (Helping Theorems)

Theorem 3.5.6

If one side of a triangle is longer than a second side, then the measure of the angle opposite the longer side is greater than the measure of the angle opposite the shorter side.

The relationship described in Theorem 3.5.6 extends, of course, to all sides and all angles of a triangle. That is, the largest of the three angles of a triangle is opposite the longest side, and the smallest angle is opposite the shortest side.

Lemmas (Helping Theorems)

Consider Figure 3.52, in which m$\angle A$ = 80° and m$\angle B$ = 40°. Compare the lengths of the sides opposite $\angle$A and $\angle$B. It appears that the longer side lies opposite the larger angle; that is, it appears that $BC > AC$.

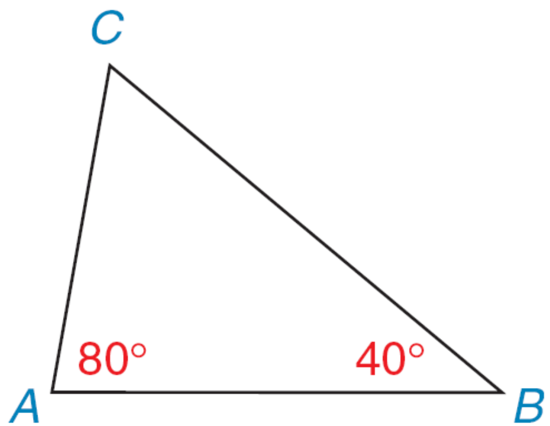

Figure 3.52

Lemmas (Helping Theorems)

Theorem 3.5.7

If the measure of one angle of a triangle is greater than the measure of a second angle, then the side opposite the larger angle is longer than the side opposite the smaller angle.

The following corollary is a consequence of Theorem 3.5.7

Corollary 3.5.8

The perpendicular line segment from a point to a line is the shortest line segment that can be drawn from the point to the line.

Lemmas (Helping Theorems)

In Figure 3.54, *PD* < *PE*, *PD* < *PF*, and *PD* < PG. In every case, $\overline{PD}$ lies opposite an acute angle of a triangle, whereas the second segment always lies opposite a right angle of that triangle (necessarily the largest angle of the triangle involved). With $\overline{PD} \perp \ell$, we say that *PD* is the *distance* from *P* to ℓ.

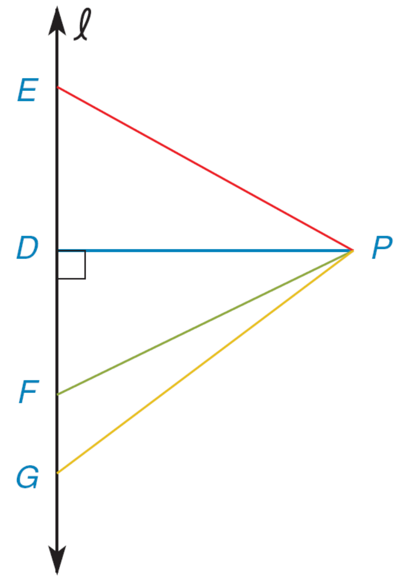

Figure 3.54

Lemmas (Helping Theorems)

Corollary 3.5.8 can easily be extended to three dimensions.

Corollary 3.5.9

The perpendicular line segment from a point to a plane is the shortest line segment that can be drawn from the point to the plane.

Lemmas (Helping Theorems)

In Figure 3.55, $\overline{PD}$ is a leg of each right triangle shown. With $\overline{PE}$ the hypotenuse of $\triangle PDE$, $\overline{PF}$ the hypotenuse of $\triangle PDF$ and $\overline{PG}$ the hypotenuse of $\triangle PDG$, the length of $\overline{PD}$ is less than that of $\overline{PE}$, $\overline{PF}$, $\overline{PG}$, or any other line segment joining point *P* to a point in plane *R*.

The length *of* $\overline{PD}$ is the *distance* from point *P* to plane *R*.

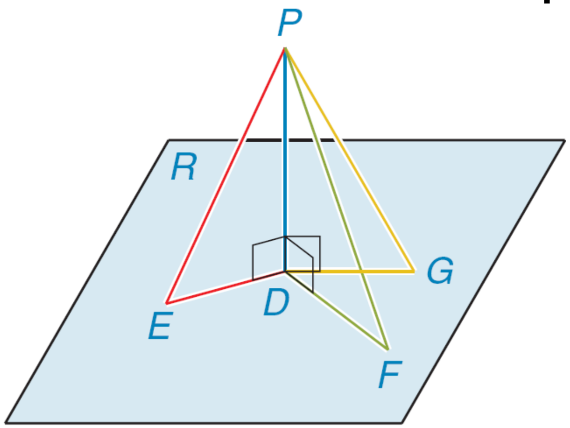

Figure 3.55

Lemmas (Helping Theorems)

Our final theorem shows that no side of a triangle can have a length greater than or equal to the sum of the lengths of the other two sides.

Theorem 3.5.10 is often called the Triangle Inequality. (See Figure 3.56.)

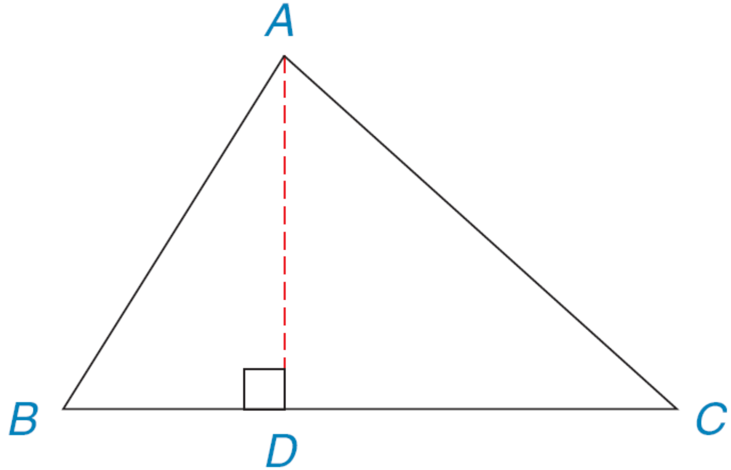

Figure 3.56

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Theorem 3.5.10 (Triangle Inequality)

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

The following statement is an alternative and expanded form of Theorem 3.5.10.

If a, b, and c are the lengths of the sides of a triangle and c is the length of any side, then $a < b + c$ and $c < a + b$; this implies that $a - b < c$ and $c < a + b$, which is equivalent to $a - b < c < a + b$.

Lemmas (Helping Theorems)

Theorem 3.5.10 (Triangle Inequality)

The length of any side of a triangle must lie between the sum and difference of the lengths of the other two sides.