Chapter 3 Triangles

3.3 Isosceles Triangles

Isosceles Triangles

In an isosceles triangle, the two sides of equal length are **legs,** and the third side is the **base.** See Figure 3.22.

The point at which the two legs meet is the **vertex** of the triangle, and the angle formed by the legs (and opposite the base) is the **vertex angle.**

The two remaining angles are **base angles.**

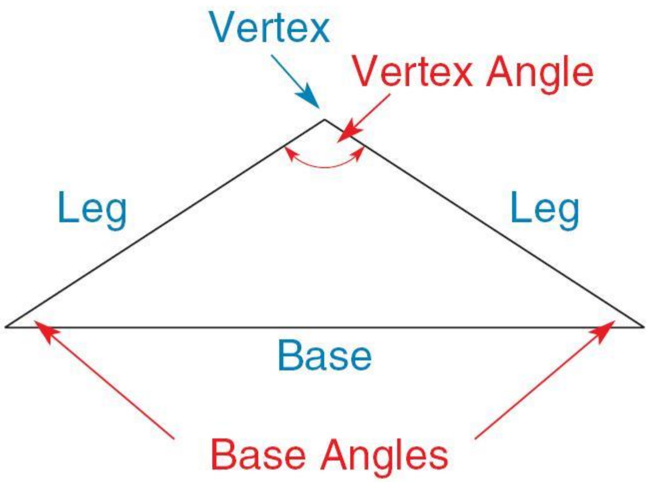


Figure 3.22

Isosceles Triangles

If $\overline{AC} \cong \overline{BC}$ in Figure 3.23, then $\triangle ABC$ is isosceles with legs $\overline{AC}$ and $\overline{BC}$, base $\overline{AB}$, vertex *C*, vertex angle *C*, and base angles at *A* and *B*.

With $\overline{AC} \cong \overline{BC}$, we see that the base $\overline{AB}$ of this isosceles triangle is not necessarily the “bottom” side.

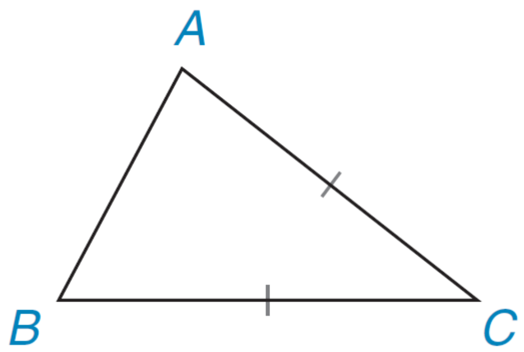
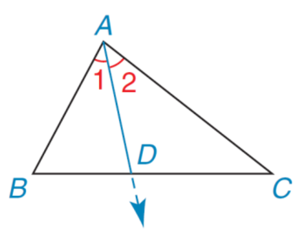


Figure 3.23

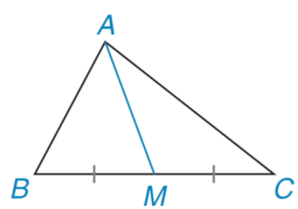
Isosceles Triangles

Helping lines known as **auxiliary lines** are needed to prove many theorems. To this end, we consider some of the lines (line segments) that may prove helpful. Each angle of a triangle has a unique **angle bisector**; this may be indicated by a ray or segment from the vertex of the bisected angle. See Figure 3.24(a).



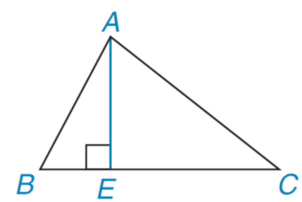
$\angle 1 \cong \angle 2$, so $\overrightarrow{AD}$ is the angle bisector of $\angle BAC$ in $\triangle ABC$

(a)



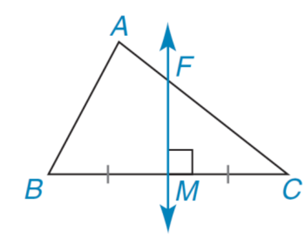
M is the midpoint of $\overline{BC}$, so $\overline{AM}$ is the median from A to $\overline{BC}$

(b)



$\overline{AE} \perp \overline{BC}$, so $\overline{AE}$ is the altitude of $\triangle ABC$ from vertex A to $\overline{BC}$

(c)



M is the midpoint of $\overline{BC}$ and $\overleftrightarrow{FM} \perp \overline{BC}$, so $\overleftrightarrow{FM}$ is the perpendicular bisector of side $\overline{BC}$ in $\triangle ABC$

(d)

Figure 3.24

Isosceles Triangles

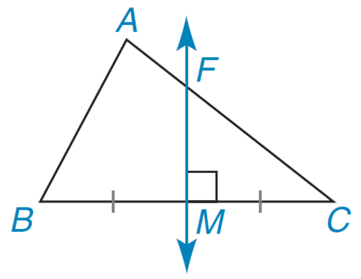
Just as an angle bisector begins at the vertex of an angle, the **median** also joins a vertex to the midpoint of the opposite side. See Figure 3.24(b).

Generally, the median from a vertex of a triangle is not the same as the angle bisector from that vertex.

An **altitude** is a line segment drawn from a vertex to the opposite side so that it is perpendicular to the opposite side. See Figure 3.24(c).

Isosceles Triangles

Finally, the **perpendicular bisector** of a side of a triangle is shown as a line in Figure 3.24(d). A segment or ray could also perpendicularly bisect a side of the triangle.



M is the midpoint of $\overline{BC}$ and $\overleftrightarrow{FM} \perp \overline{BC}$, so $\overleftrightarrow{FM}$ is the perpendicular bisector of side $\overline{BC}$ in $\triangle ABC$

(d)

Figure 3.24

Isosceles Triangles

In Figure 3.25 $\overrightarrow{AD}$ the bisector of $\angle BAC$; $\overline{AE}$ is the altitude from A to $\overline{BC}$; M is the midpoint of $\overline{BC}$; $\overline{AM}$ is the median from A to $\overline{BC}$; and $\overleftrightarrow{FM}$ is the perpendicular bisector of $\overline{BC}$.

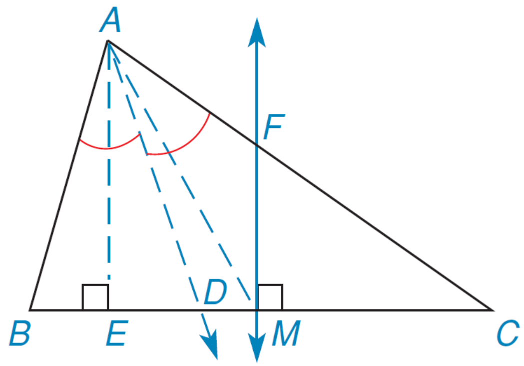


Figure 3.25

Isosceles Triangles

An altitude can actually lie in the exterior of a triangle.

In obtuse $\triangle RST$ of figure 3.26, the altitude from R must be drawn to an extension of side $\overline{ST}$.

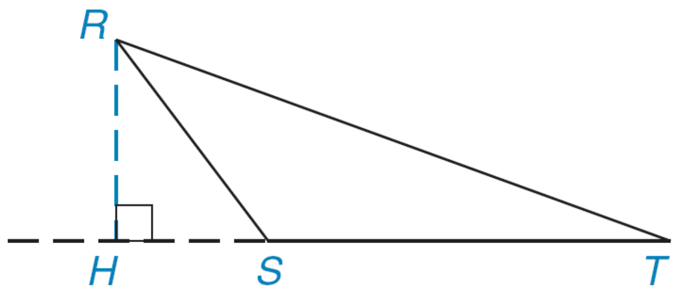


Figure 3.26

Later we will use the length h of the altitude $\overline{RH}$ and the length b of side $\overline{ST}$ in the following formula for the area of a triangle:

$$A = \frac{1}{2}bh$$

Isosceles Triangles

Any angle bisector and any median necessarily lie in the interior of the triangle.

Each triangle has three altitudes—one from each vertex. As shown for $\triangle ABC$ in Figure 3.27, the three altitudes seem to meet at a common point.

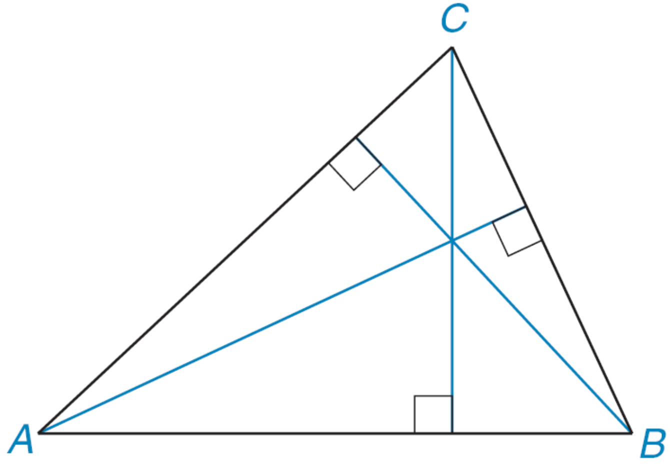


Figure 3.27

Isosceles Triangles

Theorem 3.3.1

Corresponding altitudes of congruent triangles are congruent.

Each triangle has three medians—one from each vertex to the midpoint of the opposite side.

As the medians are drawn for $\triangle DEF$ in Figure 3.28(a), it appears that the three medians intersect at a point.

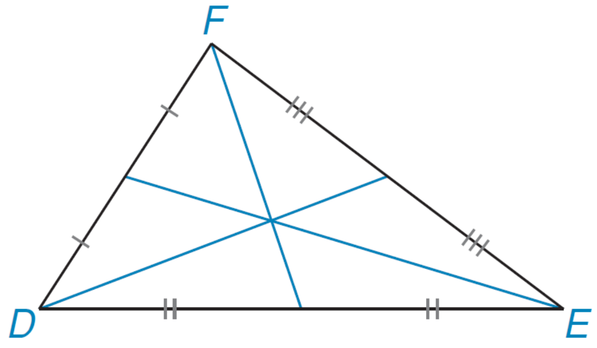


Figure 3.28(a)

Isosceles Triangles

Each triangle has three angle bisectors—one for each of the three angles.

As these are shown for $\triangle MNP$ in Figure 3.28(b), it appears that the three angle bisectors have a point in common.

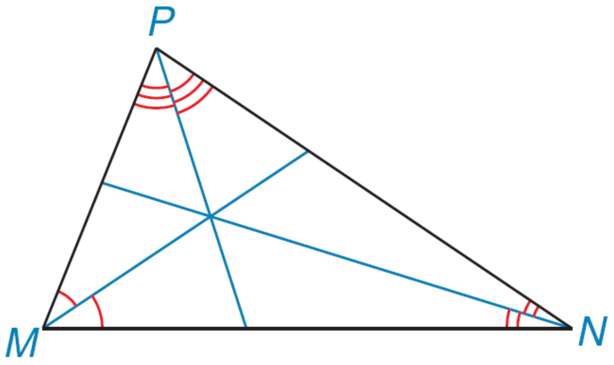


Figure 3.28(b)

Isosceles Triangles

Each triangle has three perpendicular bisectors for its sides; these are shown for $\triangle RST$ in Figure 3.28(c).

Like the altitudes, medians, and angle bisectors, the perpendicular bisectors of the sides also appear to meet at a single point.

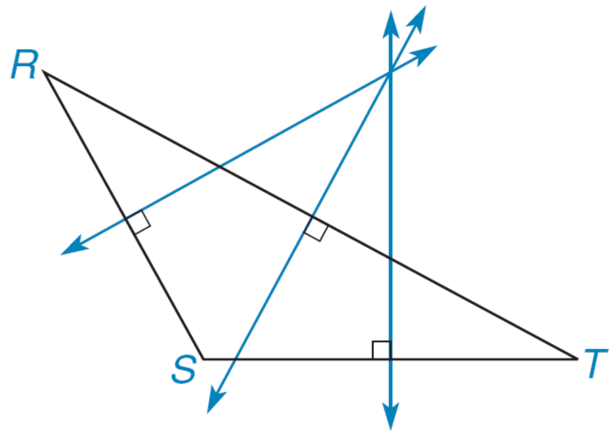


Figure 3.28(c)

Isosceles Triangles

The angle bisectors (like the medians) of a triangle *always* meet in the interior of the triangle.

However, the altitudes (like the perpendicular bisectors of the sides) can meet in the exterior of the triangle; see Figure 3.28(c).

Isosceles Triangles

In Figure 3.29, the bisector of the vertex angle of isosceles $\triangle ABC$ is a line (segment) of symmetry for $\triangle ABC$.

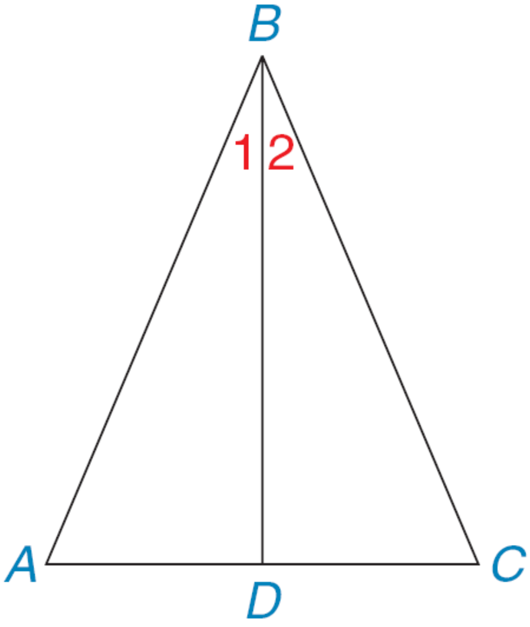


Figure 3.29

Example 1

Give a formal proof of Theorem 3.3.2.

Theorem 3.3.2

The bisector of the vertex angle of an isosceles triangle separates the triangle into two congruent triangles.

Given: Isosceles $\triangle ABC$, with $\overline{AB} \cong \overline{BC}$

$\overrightarrow{BD}$ bisects $\angle ABC$

(See Figure 3.29.)

Prove: $\triangle ABD \cong \triangle CBD$

Example 1

cont'd

Proof:

Statements	Reasons
1. Isosceles $\triangle ABC$ with $\overline{AB} \cong \overline{BC}$	1. Given
2. $\overrightarrow{BD}$ bisects $\angle ABC$	2. Given
3. $\angle 1 \cong \angle 2$	3. The bisector of an $\angle$ separates it into two $\cong \angle$ s
4. $\overline{BD} \cong \overline{BD}$	4. Identity
5. $\triangle ABD \cong \triangle CBD$	5. SAS

Isosceles Triangles

Consider Figure 3.30 and the following three descriptions, which are coded **D** for determined, **U** for underdetermined, and **O** for overdetermined:

D: Draw a line segment from *A* perpendicular to $\overline{BC}$ so that the terminal point is on $\overline{BC}$. [*Determined* because the line from *A* perpendicular to $\overline{BC}$ is unique; see Figure 3.30(a).]

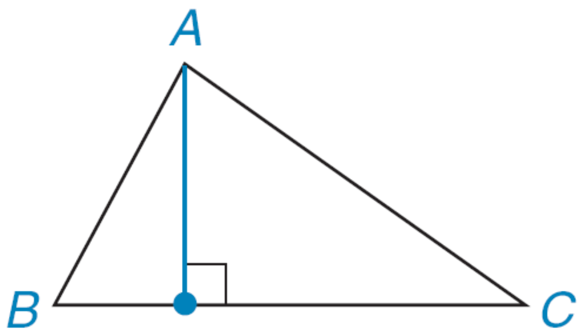


Figure 3.30 (a)

Isosceles Triangles

U: Draw a line segment from *A* to $\overline{BC}$ so that the terminal point is on $\overline{BC}$. [*Undetermined* because many line segments are possible; see Figure 3.30(b).]

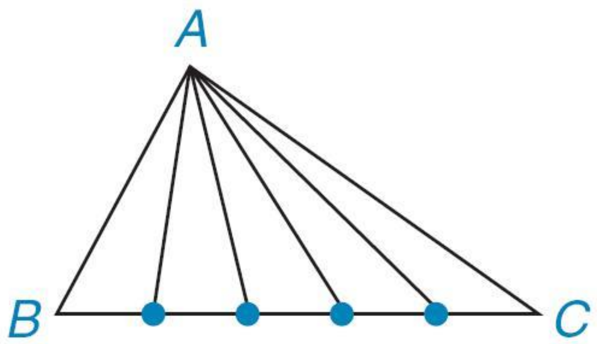


Figure 3.30 (b)

O: Draw a line segment from *A* perpendicular to $\overline{BC}$ so that it bisects $\overline{BC}$. [*Overdetermined* because the line segment from *A* drawn perpendicular to $\overline{BC}$ will not contain the midpoint *M* of $\overline{BC}$; see Figure 3.30(c).]

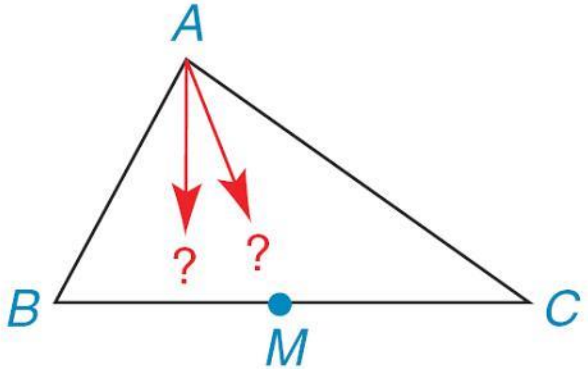


Figure 3.30 (c)

Isosceles Triangles

Theorem 3.3.3

If two sides of a triangle are congruent, then the angles opposite these sides are also congruent.

In some instances, a carpenter may want to get a quick, accurate measurement without having to go get his or her tools.

Suppose that the carpenter's square shown in Figure 3.33 is handy but that a miter box is not nearby.

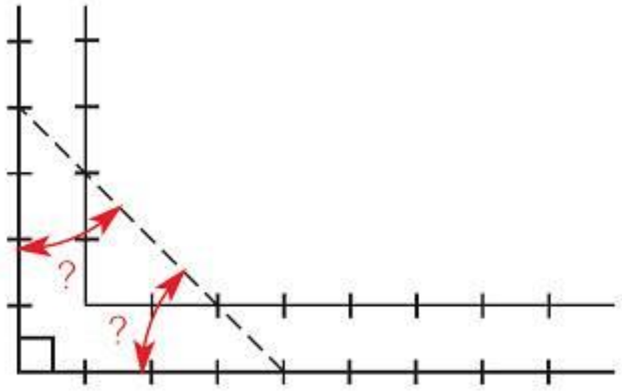


Figure 3.33

Isosceles Triangles

If two marks are made at lengths of 4 inches from the corner of the square and these are then joined, what size angle is determined? You should see that each angle indicated by an arc measures 45°.

Theorem 3.3.4

If two angles of a triangle are congruent, then the sides opposite these angles are also congruent.

Isosceles Triangles

When all three sides of a triangle are congruent, the triangle is **equilateral.**

If all three angles are congruent, then the triangle is **equiangular.**

Isosceles Triangles

Corollary 3.3.5

An equilateral triangle is also equiangular.

Corollary 3.3.6

An equiangular triangle is also equilateral.

An equilateral (or equiangular) triangle such as $\triangle XYZ$ has line symmetry with respect to each of the three axes shown in Figure 3.35.

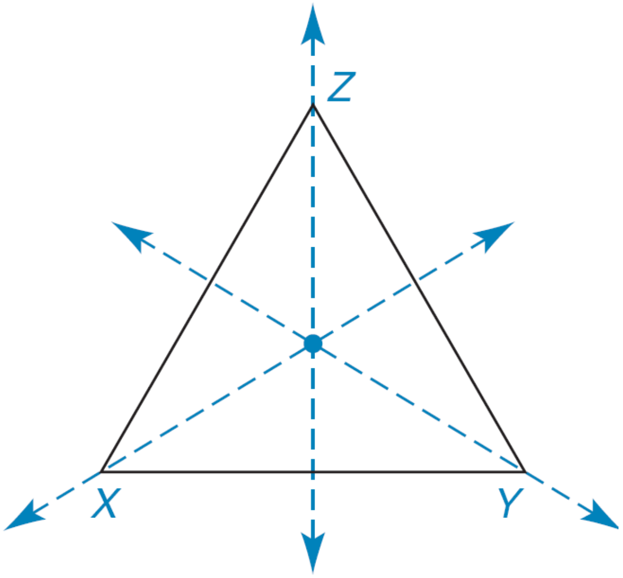


Figure 3.35

Isosceles Triangles

Definition

The **perimeter** of a triangle is the sum of the lengths of its sides. Thus, if a, b, and c are the lengths of the three sides, then the perimeter P is given by $P = a + b + c$. (See Figure 3.36.)

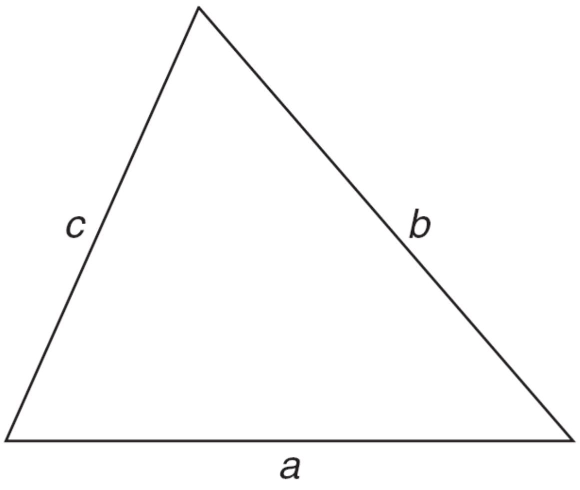
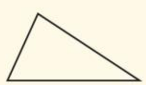
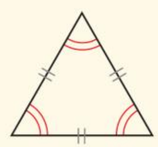
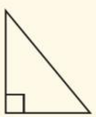


Figure 3.36

Isosceles Triangles

Many of the properties of triangles that were investigated in earlier sections of this chapter are summarized in Table 3.1.

TABLE 3.1
Selected Properties of Triangles

	Scalene	**Isosceles**	**Equilateral (equiangular)**	**Acute**	**Right**	**Obtuse**
						
Sides	No two are ≅	Exactly two are ≅	All three are ≅	Possibly two or three ≅ sides	Possibly two ≅ sides; $c^2 = a^2 + b^2$	Possibly two ≅ sides
Angles	Sum of ∠s is 180°	Sum of ∠s is 180°; two ∠s ≅	Sum of ∠s is 180°; three ≅ 60° ∠s	All ∠s acute; sum of ∠s is 180°; possibly two or three ≅ ∠s	One right ∠; sum of ∠s is 180°; possibly two ≅ 45° ∠s; acute ∠s are complementary	One obtuse ∠; sum of ∠s is 180°; possibly two ≅ acute ∠s