Chapter 3

Triangles

3.1 Congruent Triangles

Congruent Triangles

Two triangles are **congruent** if one coincides with (fits perfectly over) the other. In Figure 3.1, we say that $\triangle ABC \cong \triangle DEF$ if these congruences hold:
$\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$, $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{AC} \cong \overline{DF}$

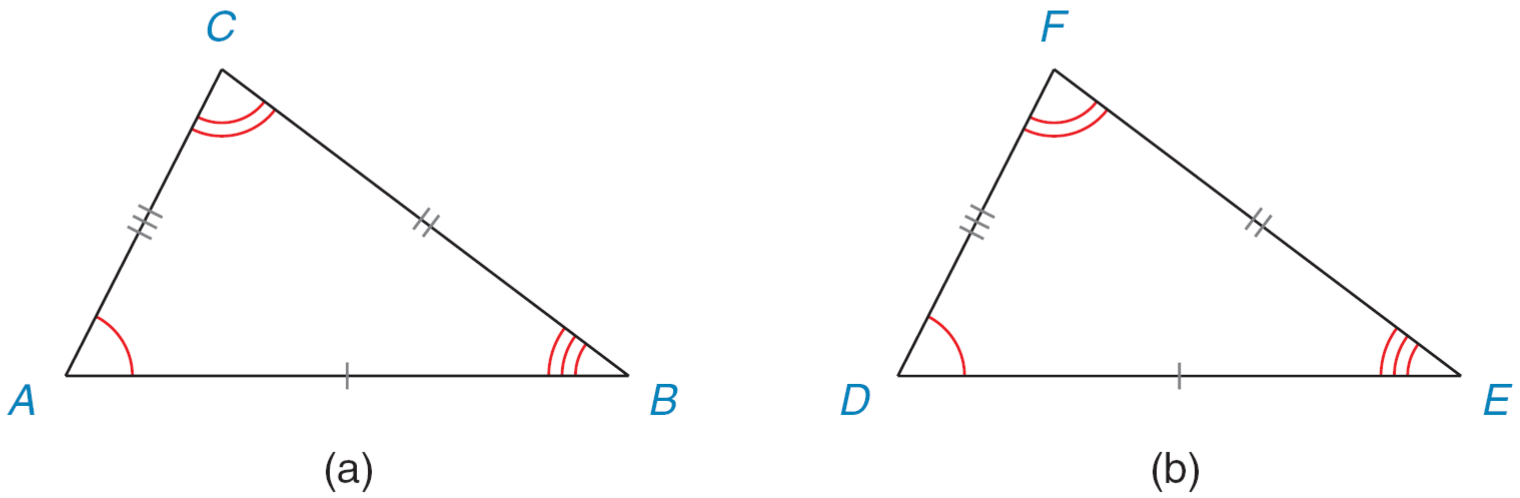

Figure 3.1

Congruent Triangles

From the indicated congruences, we also say that vertex *A* corresponds to vertex *D*, as does *B* to *E* and *C* to *F*.
In symbols, the correspondences are represented by

$$A \leftrightarrow D,\ B \leftrightarrow E, \text{ and } C \leftrightarrow F.$$

The claim $\triangle MNQ \cong \triangle RST$ orders corresponding vertices of the triangles (not shown), so we can conclude from this statement that,

$$M \leftrightarrow R,\ N \leftrightarrow S, \textit{ and } Q \leftrightarrow T.$$

Congruent Triangles

This correspondence of vertices implies the congruence of corresponding parts such as

$\angle M \cong \angle R$ and $\overline{NQ} \cong \overline{ST}$.

Conversely, if the correspondence of vertices of two congruent triangles is

$$M \leftrightarrow R,\ N \leftrightarrow S, \text{ and } Q \leftrightarrow T,$$

we order these vertices to make the claims $\triangle MNQ \cong \triangle RST$, $\triangle NQM \cong \triangle STR$, and so on.

Example 1

For two congruent triangles, the correspondence of vertices is given $A \leftrightarrow D$, $B \leftrightarrow E$, and $C \leftrightarrow F$. Complete each statement:

a) $\triangle BCA \cong ?$ **b)** $\triangle DEF \cong ?$

Solution:

With due attention to the order of corresponding vertices we have,

a) $\triangle BCA \cong \triangle EFD$ **b)** $\triangle DEF \cong \triangle ABC$

Congruent Triangles

Definition

Two triangles are **congruent** if the six parts of the first triangle are congruent to the six corresponding parts of the second triangle.

As always, any definition is reversible! If two triangles are known to be congruent, we may conclude that the corresponding parts are congruent. Moreover, if the six pairs of parts are known to be congruent, then so are the triangles!

Congruent Triangles

From the congruent parts indicated in Figure 3.2, we can conclude that $\triangle MNQ \cong \triangle RST$. $\triangle TSR$ is the reflection of $\triangle QNM$ across a vertical line (not shown) that lies midway between the two triangles.

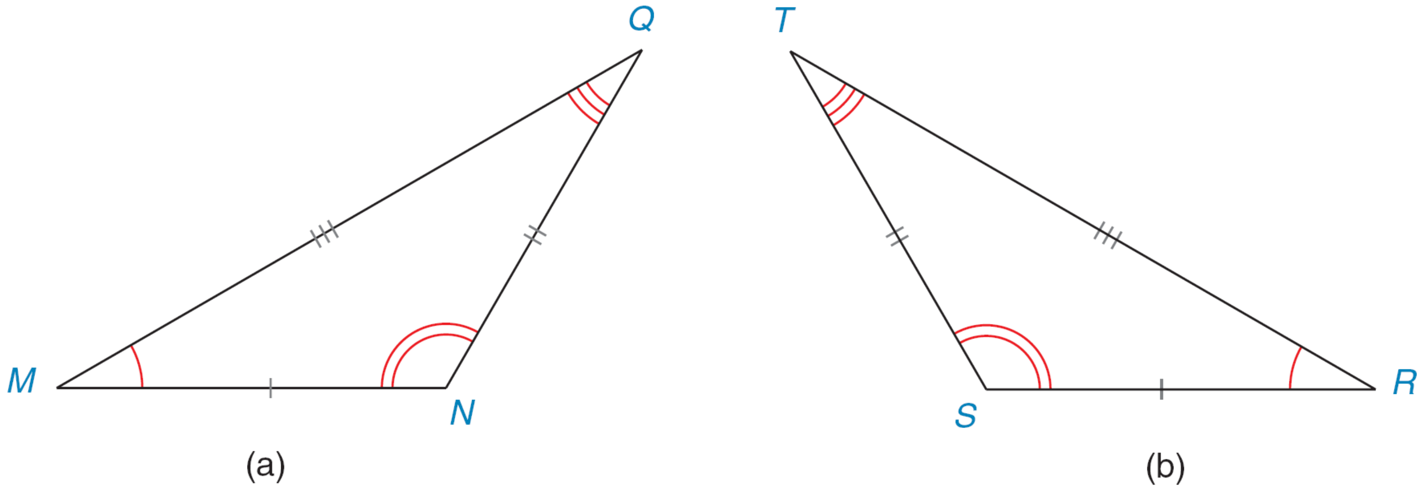

Figure 3.2

Congruent Triangles

In Figure 3.2 there are some of the properties of congruent triangles.

1. $\triangle ABC \cong \triangle ABC$ (Reflexive Property of Congruence)

2. *If* $\triangle ABC \cong \triangle DEF$, then $\triangle DEF \cong \triangle ABC$.
(Symmetric Property of Congruence)

3. If $\triangle ABC \cong \triangle DEF$ and $\triangle DEF \cong \triangle GHI$, then $\triangle ABC \cong \triangle GHI$ (Transitive Property of Congruence)

Congruent Triangles

On the basis of the properties above, we see that the "congruence of triangles" is an equivalence relation.

It would be difficult to establish that triangles were congruent if all six pairs of congruent parts had to first be verified.

Fortunately, it is possible to prove triangles congruent by establishing fewer than six pairs of congruences.

SSS (METHOD FOR PROVING TRIANGLES CONGRUENT)

SSS (Method for Proving Triangles Congruent)

Postulate 12

If the three sides of one triangle are congruent to the three sides of a second triangle, then the triangles are congruent (SSS).

The designation SSS will be cited as a reason in the proof that follows. Each letter of SSS refers to a *pair* of congruent sides.

Example 3

Given: $\overline{AB}$ and $\overline{CD}$ bisect each other at M

$\overline{AC} \cong \overline{DB}$

(See Figure 3.4.)

Prove: $\triangle AMC \cong \triangle BMD$

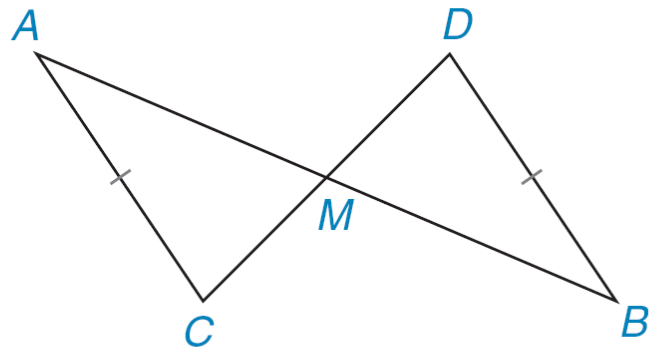

Figure 3.4

Example 3

cont'd

Proof:

Statements	**Reasons**
1. $\overline{AB}$ and $\overline{CD}$ bisect each other at *M*	1. Given
2. $\overline{AM} \cong \overline{MB}$ $\overline{CM} \cong \overline{MD}$	2. If a segment is bisected, the segments formed are ≅
3. $\overline{AC} \cong \overline{DB}$	3. Given
4. $\triangle AMC \cong \triangle BMD$	4. SSS

Example 3

cont'd

Note 1: In steps 2 and 3, the three pairs of sides were shown to be congruent; thus, SSS is cited as the reason that justifies why $\triangle AMC \cong \triangle BMD$.

Note 2: $\triangle BMD$ is the image determined by the clockwise or counterclockwise rotation of $\triangle AMC$ about point M through a $180°$ angle of rotation.

SSS (Method for Proving Triangles Congruent)

The two sides that form an angle of a triangle are said to **include that angle** of the triangle.

In $\triangle TUV$ in Figure 3.5(a), sides $\overline{TU}$ and $\overline{TV}$ form $\angle T$; therefore, $\overline{TU}$ and $\overline{TV}$ include $\angle T$. In turn, $\angle T$ is said to be the included angle for $\overline{TU}$ and $\overline{TV}$.

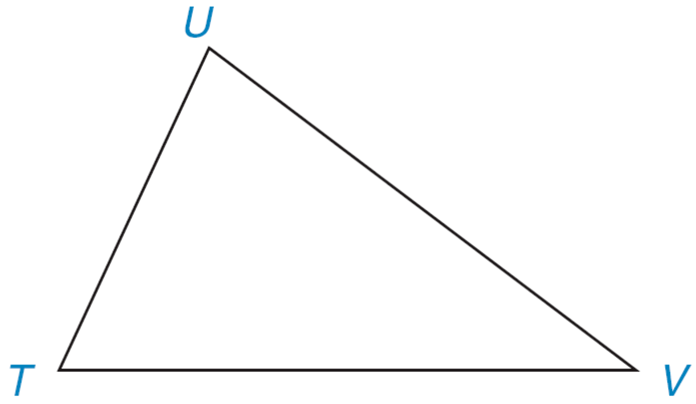

Figure 3.5(a)

SSS (Method for Proving Triangles Congruent)

Similarly, any two angles of a triangle must have a common side, and these two angles are said to **include that side**.

In $\triangle TUV$, $\angle U$ and $\angle T$ share the common side $\overline{UT}$; therefore, $\angle U$ and $\angle T$ include the side $\overline{UT}$; equivalently, $\overline{UT}$ is the side included by $\angle U$ and $\angle T$.

Informally, the term *include* names the part of a triangle that is "between" two other named parts.

SAS (METHOD FOR PROVING TRIANGLES CONGRUENT)

SAS (Method for Proving Triangles Congruent)

A second way of establishing that two triangles are congruent involves showing that two sides and the included angle of one triangle are congruent to two sides and the included angle of a second triangle.

If two people each draw a triangle so that two of the sides measure 2 cm and 3 cm and their included angle measures 54°, then those triangles are congruent. See Figure 3.6.

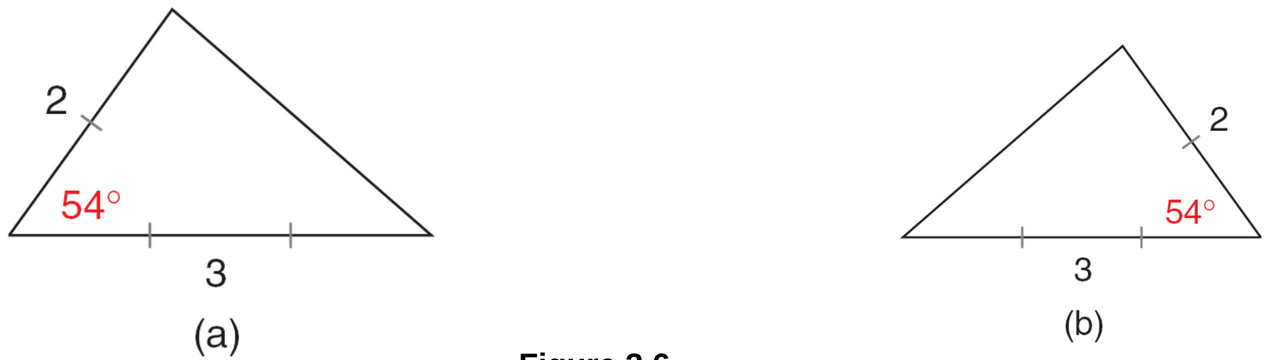

Figure 3.6

SAS (Method for Proving Triangles Congruent)

Postulate 13

If two sides and the included angle of one triangle are congruent to two sides and the included angle of a second triangle, then the triangles are congruent (SAS).

SAS (Method for Proving Triangles Congruent)

The order of the letters SAS in Postulate 13 helps us to remember that the two sides that are named have the angle “between” them; that is, the two sides referred to by S and S form the angle, represented by A.

In Example 5, which follows, the two triangles to be proved congruent share a common side; the statement $\overline{PN} \cong \overline{PN}$ is justified by the Reflexive Property of Congruence, which is conveniently expressed as **Identity**.

SAS (Method for Proving Triangles Congruent)

Definition

In a proof, **Identity** (also known as the Reflexive Property of Congruence) is the reason cited when verifying that a line segment or an angle is congruent to itself.

In Example 5, note the use of Identity and SAS as the final reasons.

Example 5

Given: $\overline{PN} \perp \overline{MQ}$

$\overline{MN} \cong \overline{NQ}$

(See Figure 3.7)

Prove: $\triangle PNM \cong \triangle PNQ$

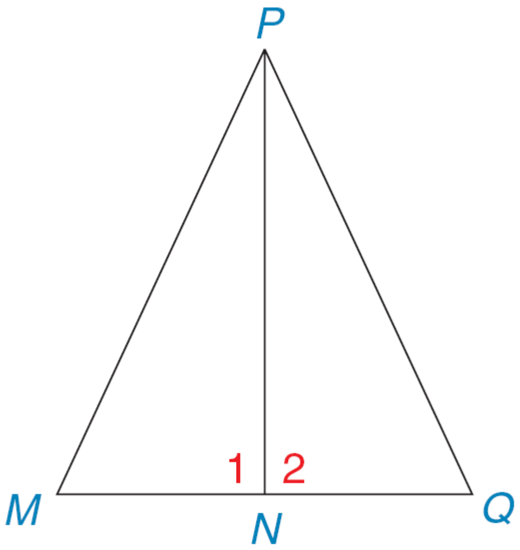

Figure 3.7

Example 5

cont'd

Proof:

Statements	Reasons
1. $\overline{PN} \perp \overline{MQ}$	1. Given
2. $\angle 1 \cong \angle 2$	2. If two lines are $\perp$, they meet to form $\cong$ adjacent $\angle$s
3. $\overline{MN} \cong \overline{NQ}$	3. Given
4. $\overline{PN} \cong \overline{PN}$	4. Identity (or Reflexive)
5. $\triangle PNM \cong \triangle PNQ$	5. SAS

ASA (METHOD FOR PROVING TRIANGLES CONGRUENT)

ASA (Method for Proving Triangles Congruent)

The next method for proving triangles congruent requires a combination of two angles and the included side.

If two triangles are drawn so that two angles measure 33° and 47° while their included side measures 5 centimeters, then these triangles must be congruent. See the figure below.

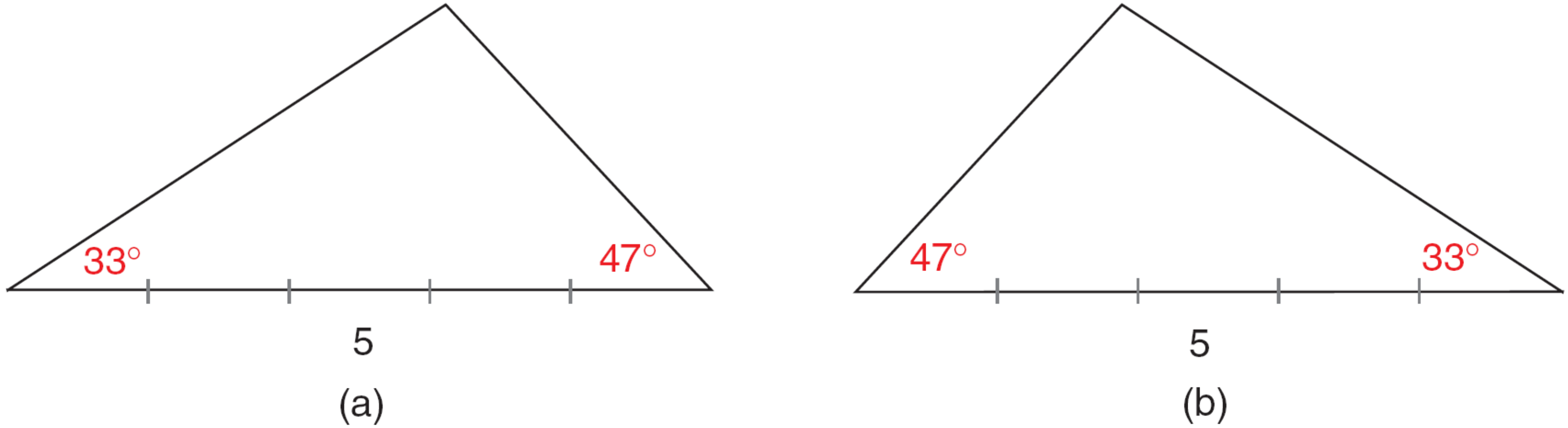

Figure 3.8

ASA (Method for Proving Triangles Congruent)

Postulate 14

If two angles and the included side of one triangle are congruent to two angles and the included side of a second triangle, then the triangles are congruent (ASA).

Although this method is written compactly as ASA, you must use caution as you write these abbreviations that verify that triangles are congruent!

ASA (Method for Proving Triangles Congruent)

For example, ASA refers to two angles and the included side, whereas SAS refers to two sides and the included angle.

For us to apply any postulate, the specific conditions described in it must be satisfied.

SSS, SAS, and ASA are all valid methods of proving triangles congruent, but SSA is *not* a method and *cannot* be used.

ASA (Method for Proving Triangles Congruent)

In Figure 3.9, the two triangles are marked to demonstrate the SSA relationship, yet the two triangles are *not* congruent.

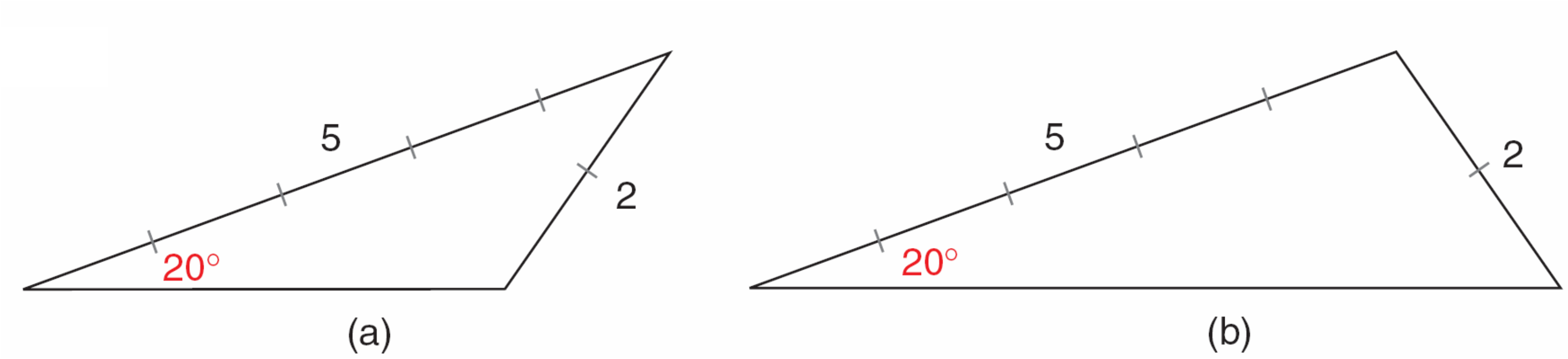

Figure 3.9

ASA (Method for Proving Triangles Congruent)

Another combination that cannot be used to prove triangles congruent is AAA. See Figure 3.10.

Three pairs of congruent angles in two triangles do not guarantee three pairs of congruent sides!

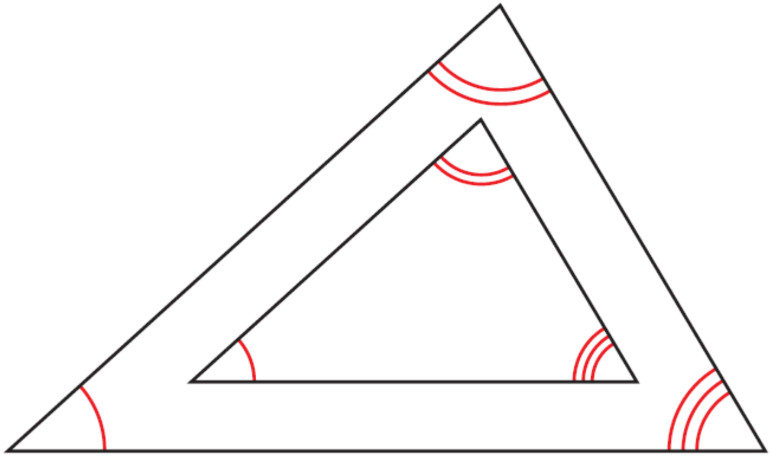

Figure 3.10

Example 6

Given: $\overline{AC} \cong \overline{DC}$

$\angle 1 \cong \angle 2$

(See Figure 3.11.)

Prove: $\triangle ACE \cong \triangle DCB$

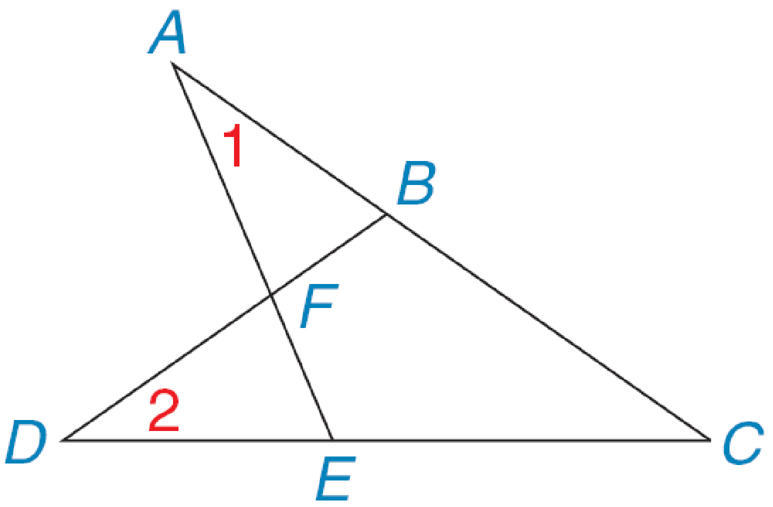

Figure 3.11

Example 6

cont'd

Proof:

Statements	Reasons
1. $\overline{AC} \cong \overline{DC}$ (See Figure 3.12.)	1. Given

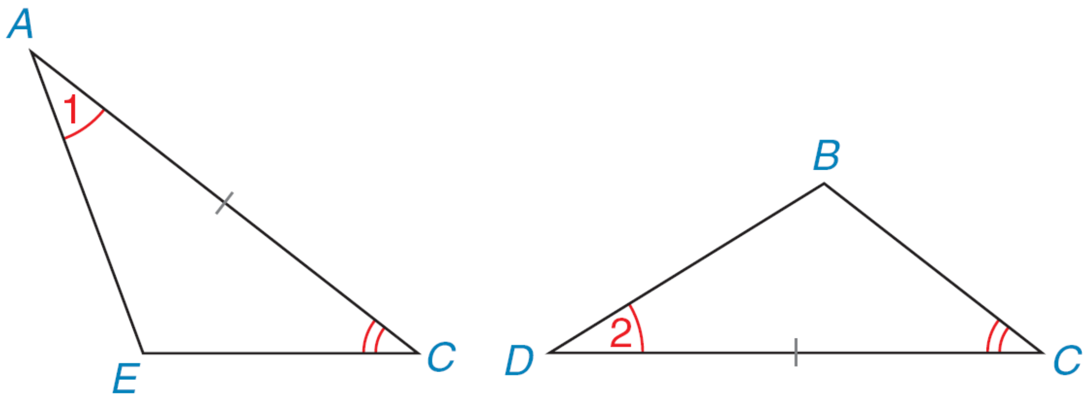

Figure 3.12

Example 6

cont'd

Statements	Reasons
2. $\angle 1 \cong \angle 2$	2. Given
3. $\angle C \cong \angle C$	3. Identity
4. $\triangle ACE \cong \triangle DCB$	4. ASA

AAS (METHOD FOR PROVING TRIANGLES CONGRUENT)

AAS (Method for Proving Triangles Congruent)

Theorem 3.1.1

If two angles and a nonincluded side of one triangle are congruent to two angles and a nonincluded side of a second triangle, then the triangles are congruent (AAS).