Chapter 11 Introduction to Trigonometry

11.4 Applications with Acute Triangles

Applications with Acute Triangles

Our focus was strictly on right triangles.

We now turn our attention to some relationships that we will prove for (and apply with) *acute* triangles.

The first relationship provides a formula for the area of a triangle in which α, β, and γ are all acute angles.

AREA OF A TRIANGLE

Area of a Triangle

Theorem 11.4.1

The area of an acute triangle equals one-half the product of the lengths of two sides and the sine of the included angle.

Theorem 11.4.1 has three equivalent forms, as shown in the following box.

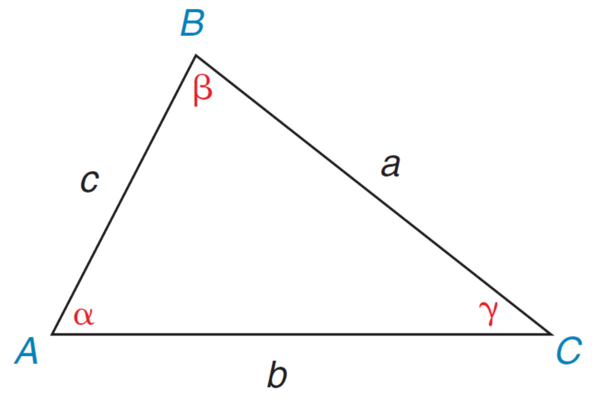
AREA OF A TRIANGLE

$$A = \frac{1}{2}bc \sin \alpha$$

Equivalently, we can prove that

$$A = \frac{1}{2}ac \sin \beta$$

$$A = \frac{1}{2}ab \sin \gamma$$



Area of a Triangle

In a more advanced study of trigonometry, the area formula found in Theorem 11.4.1 can also be proved for obtuse triangles.

In a right triangle with $\gamma = 90°$, the formula $A = \frac{1}{2}ab \sin \gamma$ reduces $A = \frac{1}{2}ab$ since $\sin \gamma = 1$.

Example 1

In Figure 11.38, find the area of $\triangle ABC$.

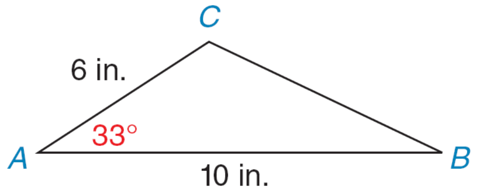


Figure 11.38

Solution:

We use the form $A = \frac{1}{2}bc \sin \alpha$, since α, b, and c are known.

$$A = \frac{1}{2} \cdot 6 \cdot 10 \cdot \sin 33°$$

$$= 30 \cdot \sin 33°$$

$$\approx 16.3 \text{ in}^2$$

LAW OF SINES

Law of Sines

Because the area of a triangle is unique, we can equate the three area expressions characterized by Theorem 11.4.1 as follows:

$$\frac{1}{2}bc\sin\alpha = \frac{1}{2}ac\sin\beta = \frac{1}{2}ab\sin\gamma$$

Multiplying by 2, $bc \sin \alpha = ac \sin \beta = ab \sin \gamma$.

Dividing each part of this equality by *abc*, we find

$$\frac{bc\sin\alpha}{abc} = \frac{ac\sin\beta}{abc} = \frac{ab\sin\gamma}{abc}$$

Law of Sines

So

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

This relationship between the lengths of the sides of an acute triangle and the sines of their opposite angles is known as the Law of Sines.

In trigonometry, it is shown that the Law of Sines is true for right triangles and obtuse triangles as well.

Law of Sines

Theorem 11.4.2 (Law of Sines)

In any acute triangle, the three ratios between the sines of the angles and the lengths of the opposite sides are equal.

That is,

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} \quad \text{or} \quad \frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

Law of Sines

When solving a problem, we equate only two of the equal ratios described in Theorem 11.4.2. For instance, we could use

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} \quad \text{or} \quad \frac{\sin \alpha}{a} = \frac{\sin \gamma}{c} \quad \text{or} \quad \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

Example 3

Use the Law of Sines to find the exact length *ST* in Figure 11.40.

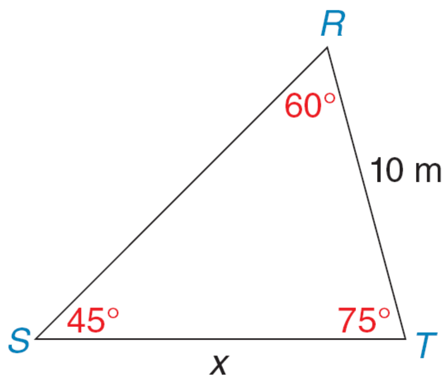


Figure 11. 40

Solution:

Because we know *RT* and the measures of angles *S* and *R*, we use $\dfrac{\sin S}{RT} = \dfrac{\sin R}{ST}$.

Example 3 – *Solution*

cont'd

Substitution of known values leads to

$$\frac{\sin 45^\circ}{10} = \frac{\sin 60^\circ}{x}$$

Because $\sin 45^\circ = \frac{\sqrt{2}}{2}$ and $\sin 60^\circ = \frac{\sqrt{3}}{2}$, we have

$$\frac{\frac{\sqrt{2}}{2}}{10} = \frac{\frac{\sqrt{3}}{2}}{x}$$

Example 3 – *Solution*

cont'd

By the Means-Extremes Property of a Proportion,

$$\frac{\sqrt{2}}{2} \cdot x = \frac{\sqrt{3}}{2} \cdot 10$$

Multiplying by 2, we have

$$\sqrt{2} \cdot x = 10\sqrt{3}$$

Then

$$x = \frac{10\sqrt{3}}{\sqrt{2}}$$

Example 3 – *Solution*

cont'd

$$= \frac{10\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{10\sqrt{6}}{2}$$

$$= 5\sqrt{6}$$

Then $ST = 5\sqrt{6}$ m.

LAW OF COSINES

Law of Cosines

The final relationship that we consider is again proved only for an acute triangle.

Like the Law of Sines, this relationship (known as the Law of Cosines) can be used to find unknown measures in a triangle.

The Law of Cosines (which can also be established for obtuse triangles in a more advanced course) can be stated in words, “The square of the length of one side of a triangle equals the sum of the squares of the lengths of the two remaining sides decreased by twice the product of the lengths of those two sides and the cosine of their included angle.”

Law of Cosines

Theorem 11.4.3 (Law of Cosines)

In acute $\triangle ABC$,

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

$$b^2 = a^2 + c^2 - 2ac \cos \beta$$

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

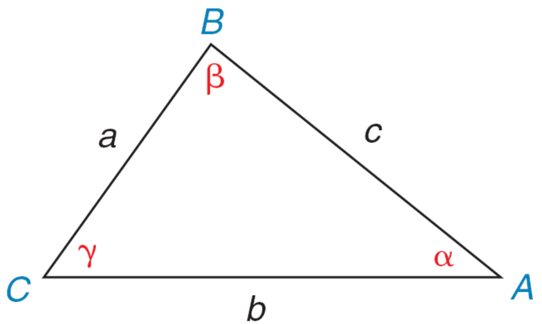


Figure 11.42

Although the Law of Cosines holds true for right triangles, the statement $c^2 = a^2 + b^2 - 2ab \cos \gamma$ reduces to the Pythagorean Theorem when $\gamma = 90°$ because $\cos 90° = 0$.

Example 5

Find the length of $\overline{AB}$ in the triangle in Figure 11.43. Then find $m\angle B$ and $m\angle A$.

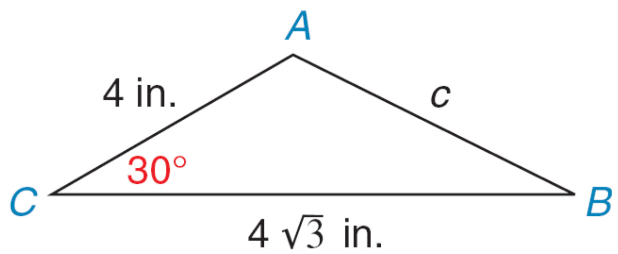


Figure 11. 43

Solution:

Referring to the 30° angle as γ, we use the following form of the Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

Example 5 – *Solution*

cont'd

$$c^2 = (4\sqrt{3})^2 + 4^2 - 2 \cdot 4\sqrt{3} \cdot 4 \cdot \cos 30^\circ$$

$$c^2 = 48 + 16 - 2 \cdot 4\sqrt{3} \cdot 4 \cdot \frac{\sqrt{3}}{2}$$

$$c^2 = 48 + 16 - 48$$

$$c^2 = 16$$

$$c = 4$$

Therefore, $AB = 4$ in.

Now $\triangle ABC$ is isosceles because $\overline{AB} \cong \overline{AC}$.

Example 5 – *Solution*

cont'd

Therefore, $\angle B \cong \angle C$. It follows that $m\angle B$ = 30° and $m\angle A$ = 120°.

Law of Cosines

The Law of Cosines can also be used to find the measure of an angle of a triangle when the lengths of its three sides are known. It is convenient to apply the following alternative form of Theorem 11.4.3.

Theorem 11.4.3 (Law of Cosines–Alternative Form)

In acute $\triangle ABC$,

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos \beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos \gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

Law of Cosines

Arguments for the remaining alternative forms are similar.

To find the measure of a side or of an angle of an acute triangle, we often have to decide which form of the Law of Sines or of the Law of Cosines should be applied.

Law of Cosines

Table 11.5 deals with that question and is based on the acute triangle shown in the accompanying drawing.

TABLE 11.5

When to Use the Law of Sines/Law of Cosines

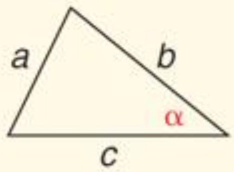
1. *Three sides are known:* Use the Law of Cosines to find *any* angle.

   Known measures: a, b, and c

   Desired measure: α

   $\therefore$ Use $a^2 = b^2 + c^2 - 2bc \cos \alpha$

   or $\cos \alpha = \dfrac{b^2 + c^2 - a^2}{2bc}$

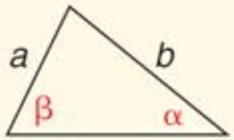


2. *Two sides and a nonincluded angle are known:* Use the Law of Sines to find the remaining nonincluded angle.

   Known measures: a, b, and α

   Desired measure: β

   $\therefore$ Use $\dfrac{\sin \alpha}{a} = \dfrac{\sin \beta}{b}$



Law of Cosines

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TABLE 11.5

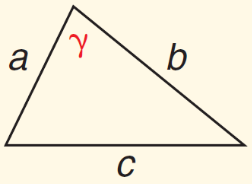
When to Use the Law of Sines/Law of Cosines

3. *Two sides and an included angle are known:* Use the Law of Cosines to find the remaining side.

Known measures: a, b, and γ

Desired measure: c

$\therefore$ Use $c^2 = a^2 + b^2 - 2ab\cos\gamma$

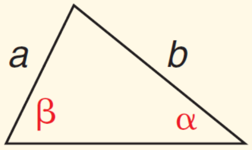


4. *Two angles and a nonincluded side are known:* Use the Law of Sines to find the other nonincluded side.

Known measures: a, α, and β

Desired measure: b

$\therefore$ Use $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b}$



Law of Cosines

Note that *a*, *b*, and *c* represent the lengths of the sides and that α, β, and γ represent the measures of the opposite angles, respectively (see Figure 11.45).

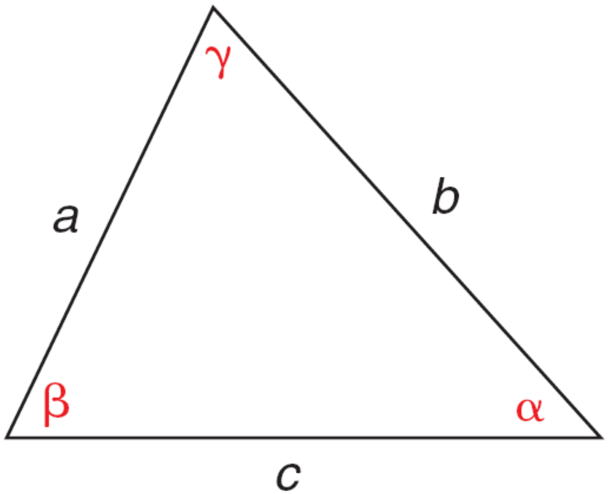


Figure 11.45