Chapter 10

Analytic Geometry

10.6 The Three-Dimensional Coordinate System

The Three-Dimensional Coordinate System

In three-dimensional space (the real world in which we live), an object can be located by its latitude, longitude, and altitude.

In mathematics, we can extend the coordinate system to include three axes; in this extension, the third axis (the z axis) is perpendicular to the xy plane (the Cartesian plane) at the point that is the common origin of all three number lines (axes).

The Three-Dimensional Coordinate System

In Figure 10.42, the three axes are mutually perpendicular, meaning that any two axes are perpendicular.

The three-dimensional coordinate system is known as *Cartesian space*; at times, the system will be called the *xyz* system.

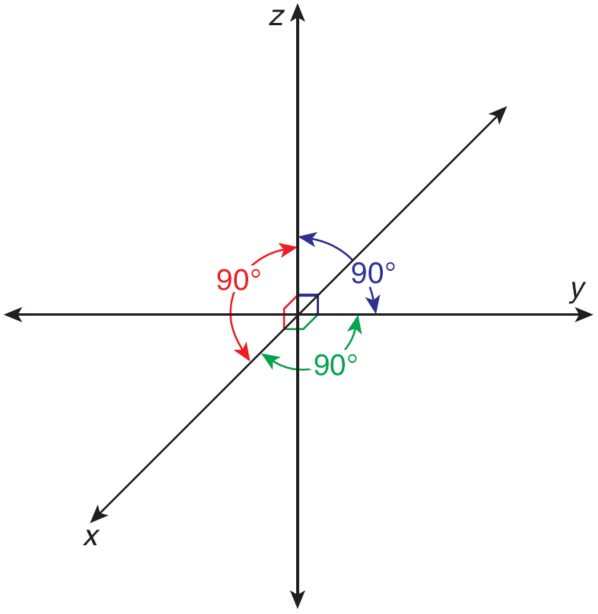


Figure 10.42

POINTS

Points

Points in the three-dimensional Cartesian system are characterized by ordered triples of the form. While the *origin* of this system is the point (0, 0, 0).

To plot a point we need to know the orientation of the axes.

In space, the *x* axis moves forward and back, the *y* axis moves right and left, and the *z* axis moves up and down.

Points

Table 10.4 indicates the positive and negative directions of the three axes.

TABLE 10.4

Point Location

	x axis	y axis	z axis
Positive direction	Forward	Right	Upward
Negative direction	Back	Left	Down

Example 1

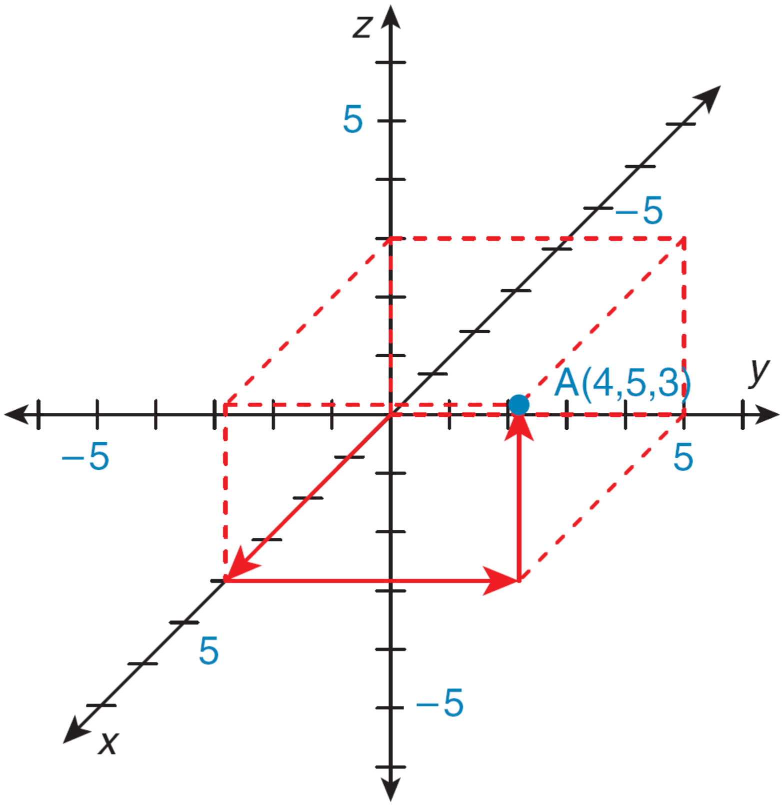
Plot each point in the three-dimensional coordinate system:

a) $A(4, 5, 3)$

b) $B(5, -6, -3)$

Solution:

a) Beginning at the origin, the point is located by moving 4 units forward, 5 units to the right, and up 3 units.



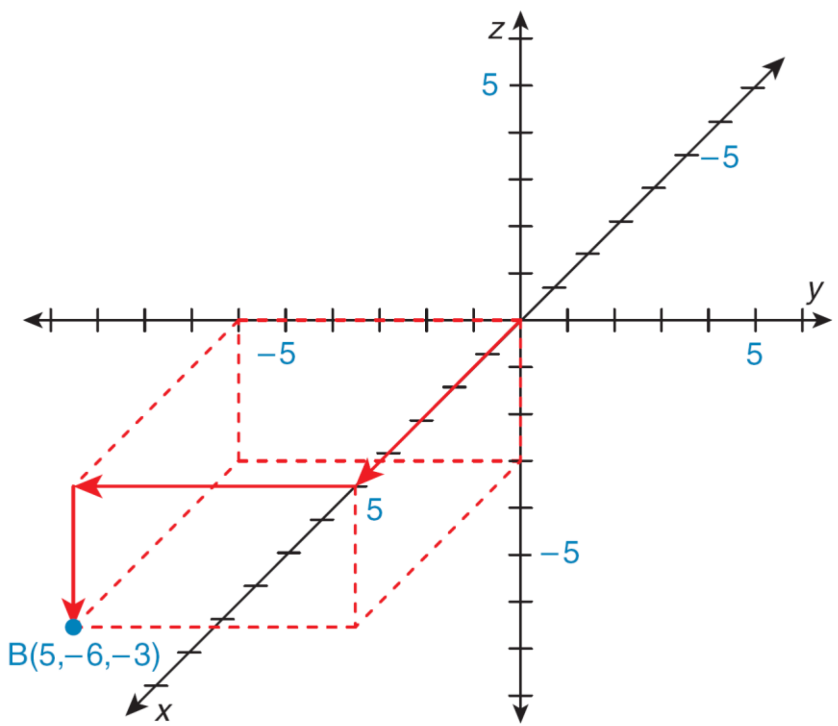
(a)

Figure 10.44

Example 1 – *Solution*

cont'd

b) Beginning at the origin, this located by moving 5 units forward, 6 units to the left, and down 3 units.



(b)

Figure 10.44

LINES

Lines

A line is determined by exactly two points in any coordinate system. Because the slope concept of the two-dimensional Cartesian system is a ratio of two numbers, there is no slope concept for a line in three dimensions.

In Cartesian space, we give direction to a line by using a *direction vector*. A direction vector of the form (a, b, c) provides changes in x, y, and z (respectively) as we trace movement along the line from one point to another point.

Lines

Definition

The line through the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ has the **direction vector** $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

Example 2

Find a direction vector for the line through (4, 5, 3) and (2, –3, –1).

Solution:

Using the definition above, a direction vector is (2 – 4, (–3) – 5, (–1) – 3) or (–2, –8, –4).

Another choice of direction vector is (2, 8, 4), found by negating the signs of the entries.

Example 2 – *Solution*

cont'd

Note: Any nonzero multiple of a direction vector of the line is also a direction vector.

For instance, multiplying the direction vectors named above by $\frac{1}{2}$ leads to (–1, –4, –2) and (1, 4, 2) as direction vectors for the line.

Lines

The equation for a line in three dimensions is actually a sum determined by a fixed point on the line and any nonzero multiple of the chosen direction vector. Before we consider the definition of the vector form of a line, consider the following operations.

Definition

The real number **multiple** of a vector is
$n(a, b, c) = (na, nb, nc)$, where n is any real number.

Also, the **sum** of two vectors (points) is
$(a, b, c) + (d, e, f) = (a + d, b + e, c + f)$.

Lines

Definition

Where (x, y, z) represents a point on the line, the **vector form** of the line through $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the equation

$$(x, y, z) = (x_1, y_1, z_1) + n(a, b, c);$$

n is any real number and $(a, b, c) = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$ is the direction vector.

The *vector form* of the line, $(x, y, z) = (x_1, y_1, z_1) + n(a, b, c)$ can be simplified and written in the form $(x, y, z) = (x_1 + na, y_1 + nb, z_1 + nc)$.

THE DISTANCE FORMULA

The Distance Formula

Some parallels between the two-dimensional coordinate system and the three-dimensional coordinate system are found in the Distance Formula and the Midpoint Formula. We state these formulas in the following theorems.

Theorem 10.6.1 (The Distance Formula)

In the *xyz* coordinate system, the distance *d* between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by

$$d = P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\,.$$

Example 4

Find the distance d between the points (5, –7, 2) and (2, 5, 6).

Solution:

When applying the formula from Theorem 10.6.1, we choose P_1 = (5, –7, 2) and P_2 = (2, 5, 6).

$$d = P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$ becomes

$$d = \sqrt{(2 - 5)^2 + (5 - [-7])^2 + (6 - 2)^2}$$

$$d = \sqrt{(-3)^2 + 12^2 + 4^2}$$

Example 4 – *Solution*

cont'd

$$d = \sqrt{9 + 144 + 16} \text{ or } \sqrt{169} \text{ or } 13.$$

That is,

$$d = P_1P_2 = 13.$$

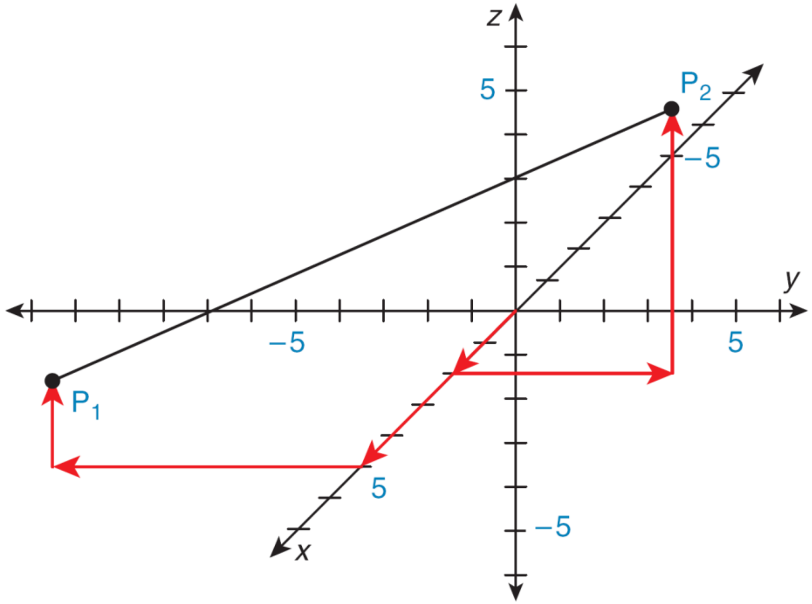


Figure 10.46

THE MIDPOINT FORMULA

The Midpoint Formula

In the Theorem 10.6.2, we characterize the midpoint by the letter M, where coordinates of the designated midpoint are $M = (x_M, y_M, z_M)$.

Theorem 10.6.2 (The Midpoint Formula)

In the xyz system, the midpoint of the line segment joining the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by

$$M = (x_M, y_M, z_M) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

Example 5

Find midpoint M of $\overline{P_1P_2}$, which joins $P_1(5, -7, 2)$ and $P_2(2, 5, 6)$ as shown in Figure 10.46.

Solution:

Applying Theorem 10.6.2 with $\overline{P_1P_2}$ of Figure 10.46, we have

$$M = \left(\frac{5+2}{2}, \frac{-7+5}{2}, \frac{2+6}{2}\right)$$

$$= \left(\frac{7}{2}, \frac{-2}{2}, \frac{8}{2}\right),$$

so $M = (3.5, -1, 4)$

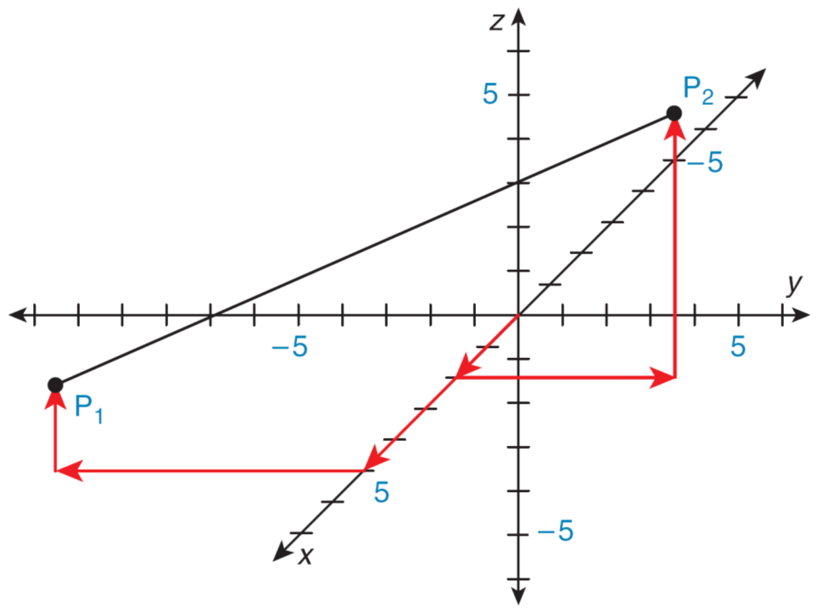


Figure 10.46

PLANES

Planes

In space, a plane is determined by three *noncollinear* points, two *intersecting* lines, or two *parallel* lines.

In space, we first consider planes that are determined by two intersecting axes. The xy plane that is determined by the intersection of the x and y axes has the equation $z = 0$.

Determined by two intersecting axes, there is an xz plane (where $y = 0$) and a yz plane (where $x = 0$). Where a, b, and c are constants, the planes that are most easily described are those of the form $x = a$, $y = b$, or $z = c$.

Planes

The graph of the linear equation $Ax + By + Cz = D$ is actually a plane.

It is still convenient to graph this plane by using *intercepts*; the *x intercept* has the form $(a, 0, 0)$ and is the point where the graph intersects the x axis.

Similarly, the *y intercept* has the form $(0, b, 0)$ while the *z intercept* has the form $(0, 0, c)$.

Example 6

Sketch the graph of the equation $x + 2y + 3z = 12$ in the *xyz* system.

Solution:

The *x* intercept is the point for which $y = 0$ and $z = 0$ thus, (12, 0, 0) is the *x* intercept.

Remaining intercepts are

(0, 6, 0) and (0, 0, 4).

The graph is shown in Figure 10.47.

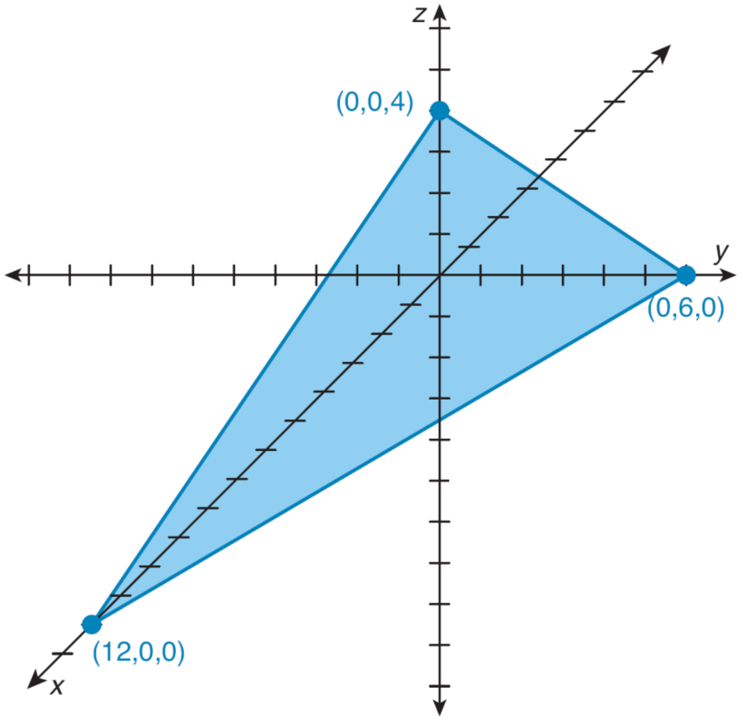


Figure 10.47

Example 6 – *Solution*

cont'd

Note:

Because (2, 2, 2) is a solution for the equation $x + 2y + 3z = 12$, the point (2, 2, 2) must lie on the plane shown.

LINE AND PLANE RELATIONSHIPS

Line and Plane Relationships

There are four possible relationships between two lines drawn in Cartesian space.

First, two lines with different direction vectors will intersect in a point if the x, y, and z coordinates of the *intersecting lines* are respectively equal.

A second possibility occurs when two lines neither move in the same direction nor intersect at a point; such lines are called *skew lines*.

Line and Plane Relationships

Third, two lines that have the same direction vectors may be *parallel* lines; for example $\ell_1 = (1, 2, 3) + n(2, 5, -6)$ and $\ell_2 = (5, 2, 7) + r(-2, -5, 6)$ are parallel because their direction vectors are multiples and the lines do *not* have a point in common.

Finally, two lines with direction vectors that are multiples coincide if they contain a common point; for instance, $\ell_1 = (1, 2, 3) + n(2, 5, -6)$ and $\ell_2 = (1, 2, 3) + r(-2, -5, 6)$ are known as *coincident lines*.

Line and Plane Relationships

Next, we illustrate a method used to determine whether two lines with different direction vectors intersect or else are skew.

Example 7

Decide if $\ell_1 = (1, 2, 3) + n(2, 5, –6)$ and $\ell_2 = (5, 7, –7) + r(1, 5, –4)$ intersect or are skew. If ℓ_1 and ℓ_2 intersect, what is the point of intersection?

Solution:

If the lines intersect, then there are values of n and r for which a point on the lines is identical.

Point forms of the lines are

$\ell_1 = (1 + 2n, 2 + 5n, 3 – 6n)$ and $\ell_2 = (5 + r, 7 + 5r, –7 – 4r)$.

Example 7 – *Solution*

cont'd

If there is a point of intersection for the two lines, then

$1 + 2n = 5 + r$, $2 + 5n = 7 + 5r$, and $3 - 6n = -7 - 4r$.

From the first equation, $r = -4 + 2n$. Substituting into the second equation,

$$2 + 5n = 7 + 5(-4 + 2n)$$

or

$$2 + 5n = 7 - 20 + 10n$$

so

$$15 = 5n \text{ and } n = 3.$$

Example 7 – *Solution*

cont'd

If $n = 3$, then $r = 2$ (from $r = –4 + 2n$); in turn, the points on the lines are

$$\ell_1 = (1 + 2n, 2 + 5n, 3 - 6n)$$

$$= (1 + 2 \cdot 3, 2 + 5 \cdot 3, 3 - 6 \cdot 3) \text{ or } (7, 17, -15),$$

and

$$\ell_2 = (5 + r, 7 + 5r, -7 - 4r)$$

$$= (5 + 2, 7 + 5 \cdot 2, -7 - 4 \cdot 2) \text{ or } (7, 17, -15).$$

Yes, the lines intersect in that the point having the common values for x and y also has a common value for z.

Example 7 – *Solution*

cont'd

Of course, the point of intersection is (7, 17, –15).

Note 1: Equating x and y produced values of n and r that could lead to different z coordinates for the two lines; in that case, the two lines are necessarily skew lines.

Note 2: The system of equations in n and r can be solved by the addition-subtraction method. If the first and third equations are used to form a system, then a common y value is sought; using the second and third equations, then a common x value is found.

Line and Plane Relationships

The four relationships for two lines in Cartesian space are summarized in Table 10.5.

TABLE 10.5

Line Relationships in Cartesian Space

ℓ_1 and ℓ_2	Direction vectors are multiples	Direction vectors are *not* multiples
have a common point	Coincide	Intersect
have no common point	Parallel	Skew

INTERSECTION OF PLANES

Intersection Of Planes

In Cartesian space, finding the line of intersection of two intersecting planes by a visual (geometric) approach is virtually impossible.

Furthermore, the algebraic technique used to determine the vector equation of the line of intersection is a real challenge.

To be complete, we examine the intersection of two nonparallel planes in Example 8.

Example 8

The intersection of the planes $x + 2y + 3z = 12$ and $2x + 3y + z = 18$ is the line given in vector form by $(x, y, z) = (0, 6, 0) + n(7, -5, 1)$.

a) Name a point on the line of intersection.

b) State a direction vector for the line of intersection.

c) Using $n = -1$ name a second point on the line of intersection.

d) Show that the point named in the solution for part (a) lies in both planes.

e) Show that the point named in the solution for part (c) lies in both planes.

Example 8 – *Solution*

a) By its vector equation form, the line of intersection must contain the point (0, 6, 0).

b) A direction vector of the line is (7, –5, 1).

c) With $n = -1$, a second point is (0, 6, 0) + –1(7, –5, 1) or (0, 6, 0) + (–7, 5, –1) or (–7, 11, –1).

Example 8 – *Solution* cont'd

d) For (0, 6, 0) to be on both planes, it must be a solution for the equation of each plane:

$x + 2y + 3z = 12$ becomes $0 + 2(6) + 3(0)$ or 12, so (0, 6, 0) is a solution;

$2x + 3y + z = 18$ becomes $2(0) + 3(6) + 0$ or 18, so (0, 6, 0) is a solution.

e) Similarly, (–7, 11, –1) must be a solution for the equation of each plane:

$x + 2y + 3z = 12$ becomes $-7 + 2(11) + 3(-1)$ or 12, so (–7, 11, –1) is a solution;

Example 8 – *Solution*

cont'd

$2x + 3y + z = 18$ becomes $2(–7) + 3(11) + (–1)$ or 18, so $(–7, 11, –1)$ is a solution.

Note: If (0, 6, 0) and (–7, 11, –1) lie on both planes then these two points determine the line of intersection for the two planes.

Intersection Of Planes

There is an alternative form for the equation of a line in three variables. Found by simplification of a real number multiple and addition, it provides a form that makes it easier to recognize points on the line.

Definition

Where (x, y, z) is any point on the line through the point $P_1(x_1, y_1, z_1)$ and having a direction vector (a, b, c), the *point form* of the line is given by

$$(x, y, z) = (x_1 + na, y_1 + nb, z_1 + nc),$$

where n is any real number.

SPHERES

Spheres

In Cartesian space, the counterpart of the circle is the sphere. To find the equation for a sphere, we apply the Distance Formula.

Where (h, k, ℓ) is the center of the sphere of radius length r, the points on the sphere (x, y, z) must lie at distance r from the center.

This relationship leads to the equation

$$\sqrt{(x - h)^2 + (y - k)^2 + (z - \ell)^2} = r.$$

Spheres

Squaring each side of the equation, we have the following theorem.

Theorem 10.6.3

The equation for the sphere with center (h, k, ℓ) and radius length r is given by the equation

$$(x - h)^2 + (y - k)^2 + (z - \ell)^2 = r^2.$$

Spheres

The *general form* for the equation of the sphere can be found by expanding the equation found in Theorem 10.6.3; thus, the general form for the equation of the sphere is

$$x^2 + y^2 + z^2 + Dx + Ey + Fz + G = 0.$$

The following corollary shows the equation for the sphere with center at the origin.

Corollary 10.6.4

The equation for the sphere with center (0, 0, 0) and radius length r is given by the equation $x^2 + y^2 + z^2 = r^2$.

Example 10

Find an equation for the sphere with

a) center at the origin and radius length $r = 5$.

b) center $(2, –3, 4)$ and radius length $r = 4$.

Solution:

a) Using Corollary 10.6.4, $x^2 + y^2 + z^2 = 5^2$, so

$$x^2 + y^2 + z^2 = 25.$$

b) Substituting into the equation in Theorem 10.6.3,

$$(x - h)^2 + (y - k)^2 + (z - \ell)^2 = r^2,$$

Example 10 – *Solution*

cont'd

we have $(x - 2)^2 + (y - [-3])^2 + (z - 4)^2 = 4^2$ or

$$(x - 2)^2 + (y + 3)^2 + (z - 4)^2 = 16$$

Expanding the equation,

$$x^2 - 4x + 4 + y^2 + 6y + 9 + z^2 - 8z + 16 = 16.$$

In general form, the equation of the sphere is

$$x^2 + y^2 + z^2 - 4x + 6y - 8z + 13 = 0.$$