Chapter 10 Analytic Geometry

10.4 Analytic Proofs

Analytic Proofs

When we use algebra along with the rectangular coordinate system to prove a geometric theorem, the method of proof is **analytic.**

Analytic Proofs

The analytic (algebraic) approach relies heavily on the placement of the figure in the coordinate system and on the application of the formulas in Table 10.1.

TABLE 10.1

Formulas of Analytic Geometry

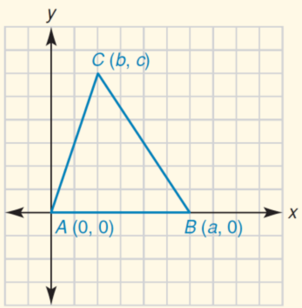
Distance	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Midpoint	$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
Slope	$m = \frac{y_2 - y_1}{x_2 - x_1}$ where $x_1 \neq x_2$
Special relationships	$\ell_1 \parallel \ell_2 \leftrightarrow m_1 = m_2$
for lines	$\ell_1 \perp \ell_2 \leftrightarrow m_1 \cdot m_2 = -1$, where neither ℓ_1 nor ℓ_2 is a vertical line or a horizontal line.

Analytic Proofs

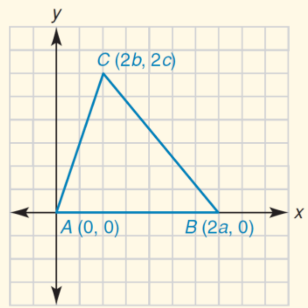
We review this information in Table 10.2.

TABLE 10.2

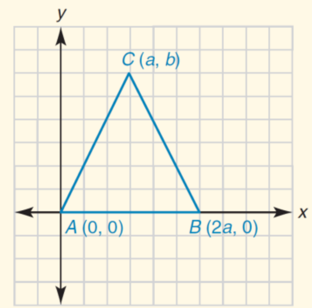
Analytic Proof: Suggestions for Placement of the Triangle



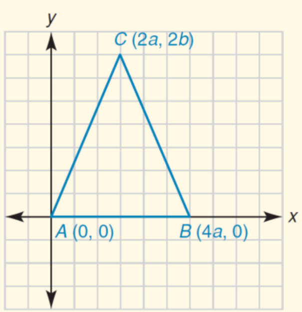
General Triangle



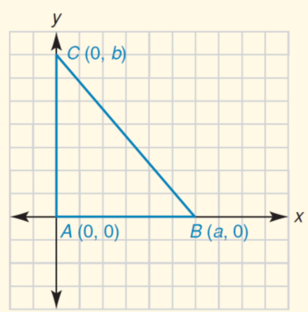
General Triangle
(Midpoints)



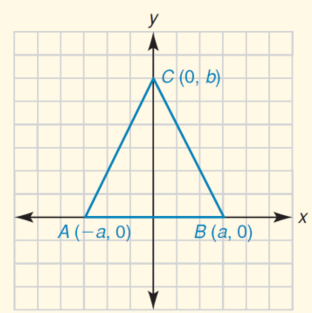
Isosceles Triangle



Isosceles Triangle
(Midpoints)



Right Triangle



Equilateral Triangle
(where $2a = \sqrt{a^2 + b^2}$, so $3a^2 = b^2$)

Analytic Proofs

In Table 10.2, you will find that the figure determined by any positive numerical choices of a, b, and c matches the type of triangle described.

When midpoints are involved, we use coordinates such as $2a$ or $2b$.

Example 1

Prove the following theorem by the analytic method (see Figure 10.29).

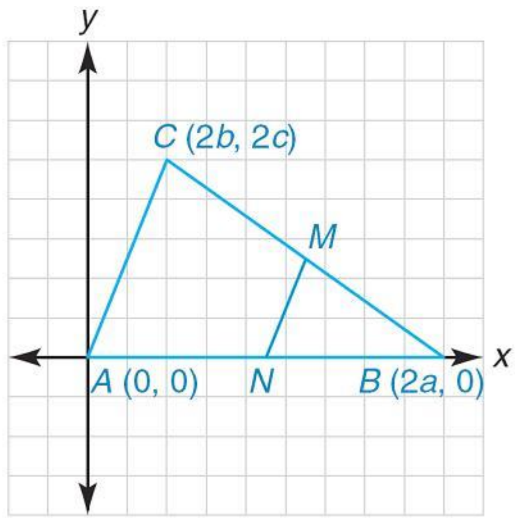


Figure 10.29

Theorem 10.4.1

The line segment determined by the midpoints of two sides of a triangle is parallel to the third side.

Example 1

Plan: Use the Slope Formula; if $m_{\overline{MN}} = m_{\overline{AC}}$, then $\overline{MN} \parallel \overline{AC}$.

Proof: As shown in Figure 10.29, $\triangle ABC$ has vertices at $A(0, 0)$, $B(2a, 0)$, and $C(2b, 2c)$. With M the midpoint of $\overline{BC}$, and N the midpoint of $\overline{AB}$,

$$M = \left(\frac{2a + 2b}{2}, \frac{0 + 2c}{2}\right), = (a + b, c)$$

$$N = \left(\frac{0 + 2a}{2}, \frac{0 + 0}{2}\right), = (a, 0)$$

Example 1

cont'd

Next we apply the Slope Formula to determine $m_{\overline{MN}}$ and $m_{\overline{AC}}$.

Now $m_{\overline{MN}} = \dfrac{c - 0}{(a + b) - a} = \dfrac{c}{b}$;

also, $m_{\overline{AC}} = \dfrac{2c - 0}{2b - 0} = \dfrac{2c}{2b} = \dfrac{c}{b}$.

Because $m_{\overline{MN}} = m_{\overline{AC}}$,

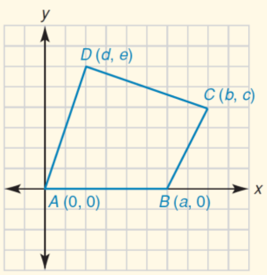
we see that $\overline{MN} \parallel \overline{AC}$.

Analytic Proofs

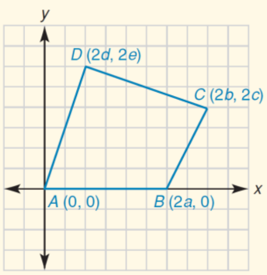
In Table 10.3, we review convenient placements for types of quadrilaterals.

TABLE 10.3

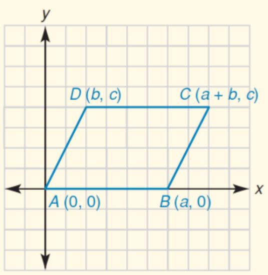
Analytic Proof: Suggestions for Placement of the Quadrilateral



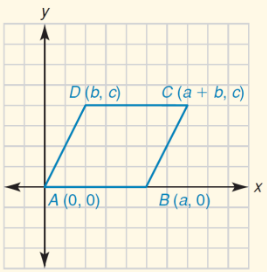
General Quadrilateral



General Quadrilateral
(Midpoints)

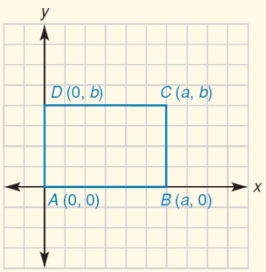


Parallelogram

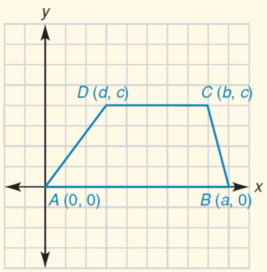


Rhombus

(where $a = \sqrt{b^2 + c^2}$, so $a^2 = b^2 + c^2$)



Rectangle



Trapezoid

Analytic Proofs

Theorem 10.4.2

The diagonals of a parallelogram bisect each other.

Theorem 10.4.3

The diagonals of a rhombus are perpendicular.

Theorem 10.4.4

If the diagonals of a parallelogram are equal in length, then the parallelogram is a rectangle.