Chapter 10

Analytic Geometry

10.2 Graphs of Linear Equations and Slope

Graphs of Linear Equations and Slope

The general form of the equation of a line is $Ax + By = C$ (where A and B do not both equal 0).

Some examples of *linear* equations are $2x + 3y = 12$, $3x - 4y = 12$, and $3x = -6$; as we shall see, the graph of each of these equations is a line.

THE GRAPH OF AN EQUATION

The Graph of an Equation

Definition

In the rectangular coordinate system, the **graph of an equation** is the set of all points (x, y) whose ordered pairs satisfy the equation.

Example 1

Draw the graph of the equation $2x + 3y = 12$.

Solution:

We begin by completing a table. It is convenient to use one point for which $x = 0$, a second point for which $y = 0$, and a third point as a check for collinearity.

$$2x + 3y = 12$$

$$x = 0 \rightarrow 2(0) + 3y = 12 \rightarrow y = 4$$

$$y = 0 \rightarrow 2x + 3(0) = 12 \rightarrow x = 6$$

$$x = 3 \rightarrow 2(3) + 3y = 12 \rightarrow y = 2$$

Example 1 – *Solution*

cont'd

x	y	(x, y)
0	4	(0, 4)
6	0	(6, 0)
3	2	(3, 2)

Upon plotting the third point, we see that the three points are collinear. The graph of a linear equation must be a straight line, as shown in Figure 10.10.

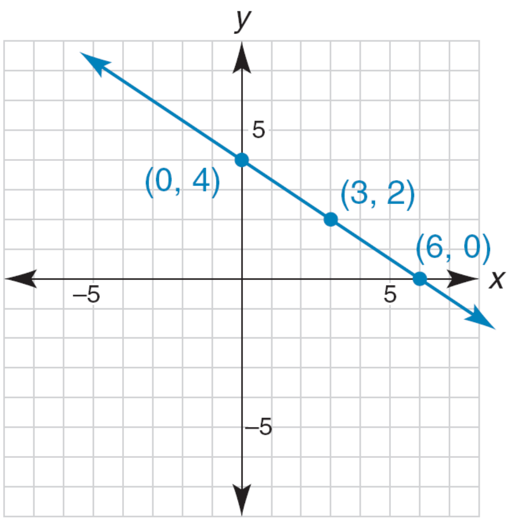

Figure 10.10

The Graph of an Equation

For the graph and equation in Example 1, every point on the line must also satisfy the given equation. Notice that the point $(–3, 6)$ lies on the line shown in Figure 10.10.

This ordered pair also satisfies the equation $2x + 3y = 12$; that is, $2(–3) + 3(6) = 12$ because $–6 + 18 = 12$.

For the equation in $2x + 3y = 12$, the number 6 is known as the ***x* intercept** because $(6, 0)$ is the point at which the graph crosses the *x* axis; similarly, the number 4 is known as the ***y* intercept**.

The Graph of an Equation

Most linear equations have two intercepts; these are generally represented by *a* (the *x* intercept) and *b* (the *y* intercept).

For the equation $Ax + By = C$, we determine the

a) x intercept by choosing $y = 0$. Solve the resulting equation for x.
b) y intercept by choosing $x = 0$. Solve the resulting equation for y.

THE SLOPE OF A LINE

The Slope of a Line

Most lines are oblique; that is, the line is neither horizontal nor vertical. Especially for oblique lines, it is convenient to describe the amount of "slant" by a number called the *slope* of the line.

Definition (Slope Formula)

The slope of the line that contains the points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{for } x_1 \neq x_2$$

The Slope of a Line

Note: When $x_1 = x_2$, the denominator of the Slope Formula becomes 0 and we say that the slope of the line is undefined.

Whereas the uppercase italic *M* means midpoint, we use the lowercase italic *m* to represent the slope of a line. Other terms that are used to describe the slope of a line include *pitch* and *grade*.

Example 5

Without graphing, find the slope of the line that contains:

a) (2, 2) and (5, 3) b) (1, –1) and (1, 3)

Solution:

a) Using the Slope Formula and choosing $x_1 = 2$, $y_1 = 2$, $x_2 = 5$, and $y_2 = 3$, we have

$$m = \frac{3 - 2}{5 - 2}$$

$$= \frac{1}{3}$$

Example 5 – *Solution*

cont'd

b) Let $x_1 = 1$, $y_1 = –1$, $x_2 = 1$, and $y_2 = 3$.

Then we calculate

$$m = \frac{3 - (-1)}{1 - 1}$$

$$= \frac{4}{0}$$

which is undefined.

The Slope of a Line

Theorem 10.2.1

If two nonvertical lines are parallel, then their slopes are equal.

Alternative Form: If $\ell_1 \parallel \ell_2$, then $m_1 = m_2$.

In Figure 10.18, note that $\overline{AC} \parallel \overline{DF}$. Also, $\overline{AB}$ and $\overline{DE}$ are horizontal, and $\overline{BC}$ and $\overline{EF}$ are auxiliary vertical segments.

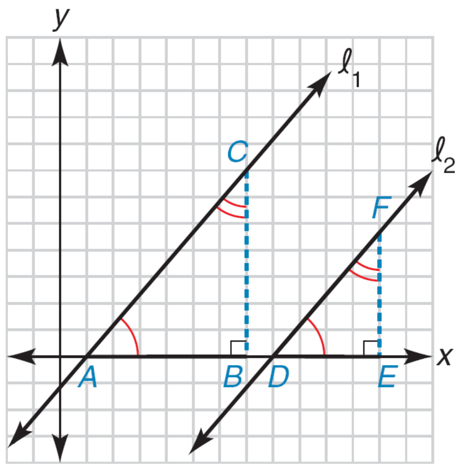

Figure 10.18

The converse of Theorem 10.2.1 is also true; that is, if $m_1 = m_2$, then $\ell_1 \parallel \ell_2$.

The Slope of a Line

Theorem 10.2.2

If two lines (neither horizontal nor vertical) are perpendicular, then the product of their slopes is –1.

Alternative Form: If $\ell_1 \perp \ell_2$, then $m_1 \cdot m_2 = -1$.

In Figure 10.19, auxiliary segments have been included.

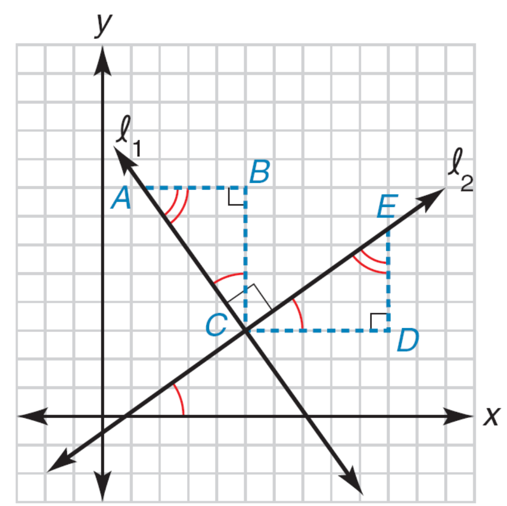

Figure 10.19

The Slope of a Line

Because the product of the slopes is –1, the slopes are **negative reciprocals**. In general, negative reciprocals take the forms $\frac{a}{b}$ and $\frac{-b}{a}$.

The converse of Theorem 10.2.2 is also true; if $m_1 \cdot m_2 = -1$, then $\ell_1 \perp \ell_2$.