Chapter 11 Introduction to Trigonometry

11.1 The Sine Ratio and Applications

The Sine Ratio and Applications

In this section, we deal exclusively with similar right triangles.

In Figure 11.1, $\triangle ABC \sim \triangle DEF$ and $\angle C$ and $\angle F$ are right angles.

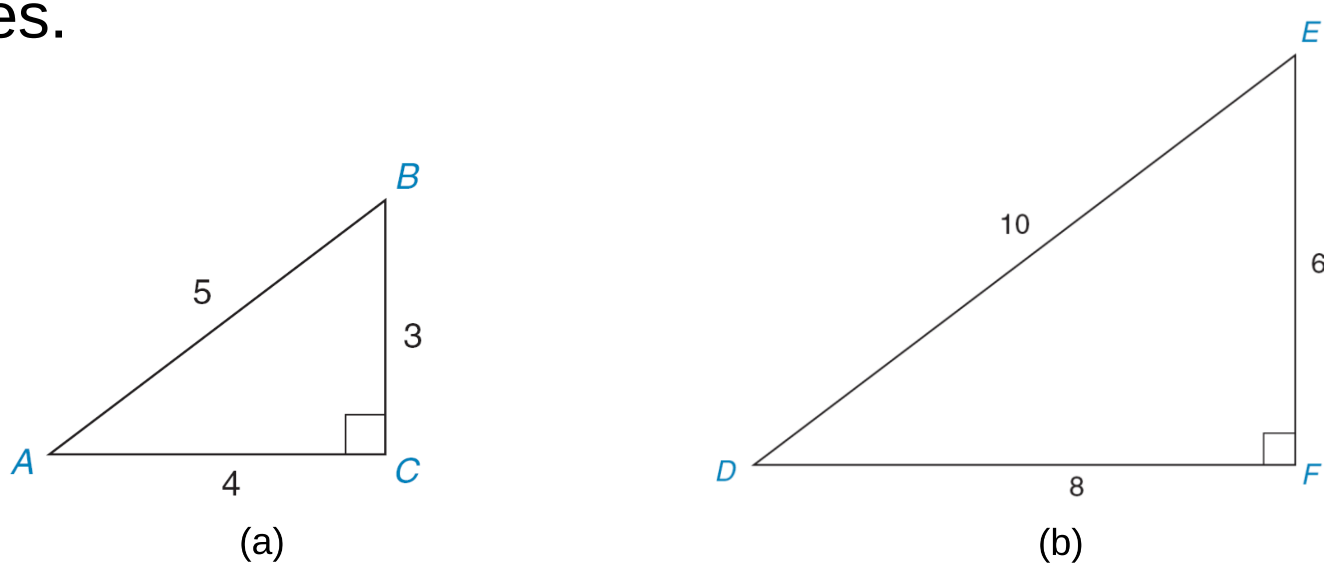


Figure 11.1

The Sine Ratio and Applications

Consider corresponding angles *A* and *D*; if we compare the length of the side opposite each angle to the length of the hypotenuse of each triangle, we obtain this result by the reason CSSTP:

$$\frac{BC}{AB} = \frac{EF}{DE} \quad \text{or} \quad \frac{3}{5} = \frac{6}{10}$$

The Sine Ratio and Applications

In the two similar right triangles, the ratio of this pair of corresponding sides depends on the measure of acute $\angle A$ (or $\angle D$, because $m\angle A = m\angle D$); for each angle, the numerical value of the ratio is unique.

$$\frac{\text{length of side opposite the acute angle}}{\text{length of hypotenuse}}$$

This ratio becomes smaller for smaller measures of $\angle A$ and larger for larger measures of $\angle A$.

The Sine Ratio and Applications

This ratio is unique for each measure of an acute angle even though the lengths of the sides of the two similar right triangles containing the angle are different.

In Figure 11.2, we name the measures of the angles of the right triangle by the Greek letters α (alpha) at vertex *A*, β (beta) at vertex *B*, and γ (gamma) at vertex *C*.

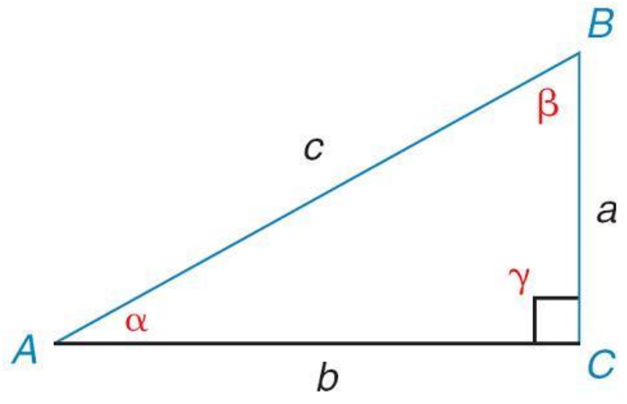


Figure 11.2

The Sine Ratio and Applications

The lengths of the sides opposite vertices A, B, and C are a, b, and c, respectively.

Relative to the acute angle, the lengths of the sides of the right triangle in the following definition are described as “opposite” and “hypotenuse.”

The word **opposite** is used to mean the length of the side opposite the angle named; the word **hypotenuse** is used to mean the length of the hypotenuse.

The Sine Ratio and Applications

Definition

In a right triangle, the **sine ratio** for an acute angle is the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$.

Example 1

In Figure 11.3, find sin α and sin β for right $\triangle ABC$.

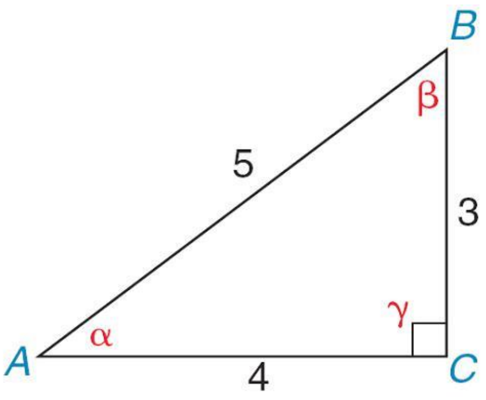


Figure 11.3

Solution:

$a = 3$, $b = 4$, and $c = 5$.

Therefore,

$$\sin\alpha = \frac{a}{c} = \frac{3}{5}$$

and

$$\sin\beta = \frac{b}{c} = \frac{4}{5}$$

The Sine Ratio and Applications

Where α is the measure of an acute angle of a right triangle, the value of sin α is unique.

Although the sine ratios for angle measures are readily available on a calculator, we can justify several of the calculator's results by using special triangles.

For certain angles, we can find *exact* results, whereas the calculator provides approximations.

The Sine Ratio and Applications

We know the 30°-60°-90° relationship, in which the side opposite the 30° angle has a length equal to one-half that of the hypotenuse; the remaining leg has a length equal to the product of the length of the shorter leg and $\sqrt{3}$.

Example 3

Find exact and approximate values for sin 30° and sin 60°. See Figure 11.5.

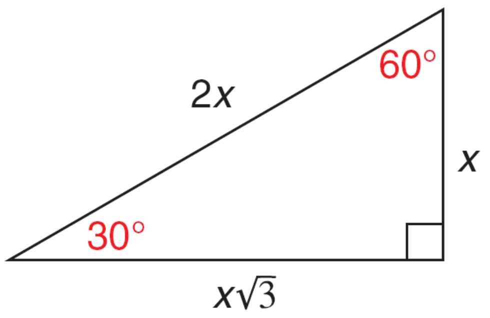


Figure 11.5

Solution:

Using the 30°-60°-90° triangle shown in Figure 11.5,

$$\sin 30^\circ = \frac{x}{2x} = \frac{1}{2}$$

while

$$\sin 60^\circ = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2} \approx 0.866$$

The Sine Ratio and Applications

Although the exact value of sin 30° is 0.5 and the exact value of sin 60° is $\frac{\sqrt{3}}{2}$, a calculator would give an approximate value for sin 60° such as 0.8660254.

If we round the ratio for sin 60° to four decimal places, then sin 60° ≈ 0.8660. Use your calculator to show that $\frac{\sqrt{3}}{2} \approx 0.8660$ as well.

We will now use the Angle-Bisector Theorem to determine the sine ratios for angles that measure 15° and 75°.

The Sine Ratio and Applications

We know that the bisector of one angle of a triangle divides the opposite side into two segments that are proportional to the sides forming the bisected angle.

Using the resulting triangle shown in Figure 11.7, we are led to the proportion

$$\frac{y}{1 - y} = \frac{\sqrt{3}}{2}$$

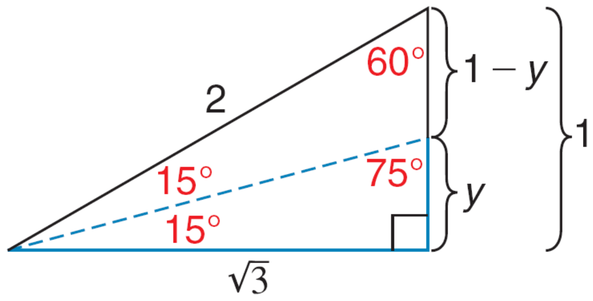


Figure 11.7

The Sine Ratio and Applications

Applying the Means-Extremes Property, we have

$$\begin{aligned} 2y &= \sqrt{3} - y\sqrt{3} \\ 2y + y\sqrt{3} &= \sqrt{3} \\ (2 + \sqrt{3})y &= \sqrt{3} \\ y &= \frac{\sqrt{3}}{2 + \sqrt{3}} \approx 0.4641 \end{aligned}$$

The number 0.4641 is the length of the side that is opposite the 15° angle of the 15°-75°-90° triangle in Figure 11.8.

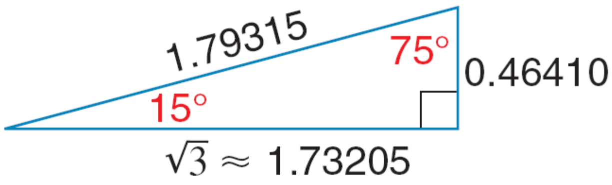


Figure 11.8

The Sine Ratio and Applications

Using the Pythagorean Theorem, we can show that the length of the hypotenuse is approximately 1.79315.

In turn, sin 15° $= \frac{0.46410}{1.79315}$ ≈ 0.2588.

Using the same triangle, we get sin 75° $= \frac{1.73205}{1.79315}$ ≈ 0.9659.

The Sine Ratio and Applications

We now begin to formulate a small table of values of sine ratios. In Table 11.1, the Greek letter θ (theta) designates the angle measure in degrees.

The second column has the heading sin θ and provides the ratio for the corresponding angle; this ratio is generally given to four decimal places of accuracy.

Note that the values of sin θ increase as θ increases in measure.

TABLE 11.1

Sine Ratios

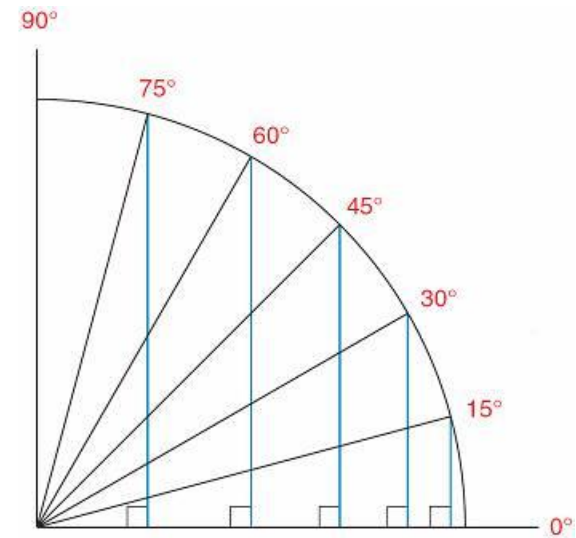
θ	$\sin\theta$
15°	0.2588
30°	0.5000
45°	0.7071
60°	0.8660
75°	0.9659

The Sine Ratio and Applications

In each right triangle shown in Figure 11.9, let $\angle\theta$ an acute angle with a side that is horizontal. In the figure, note that the length of the hypotenuse is constant—it is always equal to the length of the radius of the circle.

However, the side opposite $\angle\theta$ gets larger as θ increases in measure.

In fact, as θ approaches 90° ($\theta \rightarrow 90°$), the length of the leg opposite $\angle\theta$ approaches the length of the hypotenuse.



$\text{Sin } \theta = \frac{\text{opposite}}{\text{hypotenuse}}$ in each right triangle

Figure 11.9

The Sine Ratio and Applications

As $\theta \rightarrow 90°$, sin $\theta \rightarrow 1$. As θ decreases, sin θ also decreases. As θ decreases ($\theta \rightarrow 0°$), the length of the side opposite $\angle\theta$ approaches 0. As $\theta \rightarrow 0°$, sin $\theta \rightarrow 0$.

These observations lead to the following definition.

Definition

sin 0° = 0 and sin 90° = 1

Example 5

Using Table 11.1, find the length of *a* in Figure 11.10 to the nearest tenth of an inch.

TABLE 11.1

Sine Ratios

θ	$\sin\theta$
15°	0.2588
30°	0.5000
45°	0.7071
60°	0.8660
75°	0.9659

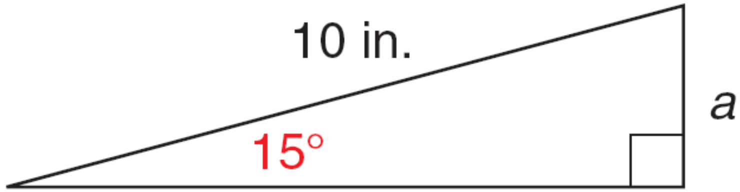


Figure 11.10

Example 5 – *Solution*

$$\sin 15° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{10}$$

From the table, we have sin 15° = 0.2588.

$$\frac{a}{10} = 0.2588 \qquad \text{(by substitution)}$$

$$a = 2.588$$

Therefore, $a \approx 2.6$ in. when rounded to tenths.

The Sine Ratio and Applications

In an application problem, the sine ratio can be used to find the measure of either a side or an angle of a right triangle.

To find the sine ratio of the angle involved, you may use a table of ratios or a calculator. Table 11.2 provides an expanded list of sine ratios; for each angle measure θ, the sine ratio is found to its immediate right.

As with calculators, the sine ratios found in tables are only approximations.

The Sine Ratio and Applications

TABLE 11.2

Sine Ratios

θ	$\sin\theta$	θ	$\sin\theta$	θ	$\sin\theta$	θ	$\sin\theta$
0°	0.0000	23°	0.3907	46°	0.7193	69°	0.9336
1°	0.0175	24°	0.4067	47°	0.7314	70°	0.9397
2°	0.0349	25°	0.4226	48°	0.7431	71°	0.9455
3°	0.0523	26°	0.4384	49°	0.7547	72°	0.9511
4°	0.0698	27°	0.4540	50°	0.7660	73°	0.9563
5°	0.0872	28°	0.4695	51°	0.7771	74°	0.9613
6°	0.1045	29°	0.4848	52°	0.7880	75°	0.9659
7°	0.1219	30°	0.5000	53°	0.7986	76°	0.9703
8°	0.1392	31°	0.5150	54°	0.8090	77°	0.9744
9°	0.1564	32°	0.5299	55°	0.8192	78°	0.9781
10°	0.1736	33°	0.5446	56°	0.8290	79°	0.9816
11°	0.1908	34°	0.5592	57°	0.8387	80°	0.9848
12°	0.2079	35°	0.5736	58°	0.8480	81°	0.9877
13°	0.2250	36°	0.5878	59°	0.8572	82°	0.9903
14°	0.2419	37°	0.6018	60°	0.8660	83°	0.9925
15°	0.2588	38°	0.6157	61°	0.8746	84°	0.9945
16°	0.2756	39°	0.6293	62°	0.8829	85°	0.9962
17°	0.2924	40°	0.6428	63°	0.8910	86°	0.9976
18°	0.3090	41°	0.6561	64°	0.8988	87°	0.9986
19°	0.3256	42°	0.6691	65°	0.9063	88°	0.9994
20°	0.3420	43°	0.6820	66°	0.9135	89°	0.9998
21°	0.3584	44°	0.6947	67°	0.9205	90°	1.0000
22°	0.3746	45°	0.7071	68°	0.9272		

The Sine Ratio and Applications

In most application problems, a drawing provides a good deal of information and affords some insight into the method of solution.

For some drawings and applications, the phrases *angle of elevation* and *angle of depression* are used.

The Sine Ratio and Applications

These angles are measured from the horizontal as illustrated in Figures 11.11(a) and 11.11(b).

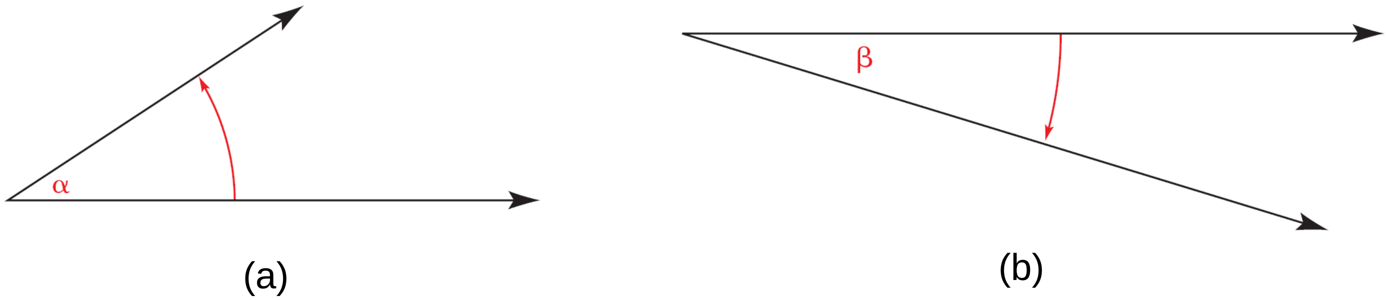


Figure 11.11

In Figure 11.11(a), the angle α measured upward from the horizontal ray is the **angle of elevation**.

In Figure 11.11(b), the angle β measured downward from the horizontal ray is the **angle of depression**.