Chapter 10 Analytic Geometry

10.1 The Rectangular Coordinate System

The Rectangular Coordinate System

Graphing the solution sets for $3x - 2 = 7$ and $3x - 2 > 7$ requires a single number line to indicate the value of x.

In this chapter, we deal with equations containing two variables; to relate such algebraic statements to plane geometry, we will need two number lines.

The study of the relationships between number pairs and points is known as **analytic geometry.**

The Rectangular Coordinate System

The **Cartesian coordinate system** or **rectangular coordinate system** is the plane that results when two number lines intersect perpendicularly at the origin (the point corresponding to the number 0 of each line).

The horizontal number line is known as the ***x* axis,** and its numerical coordinates increase from left to right.

On the vertical number line, the ***y* axis,** values increase from bottom to top; see Figure 10.1.

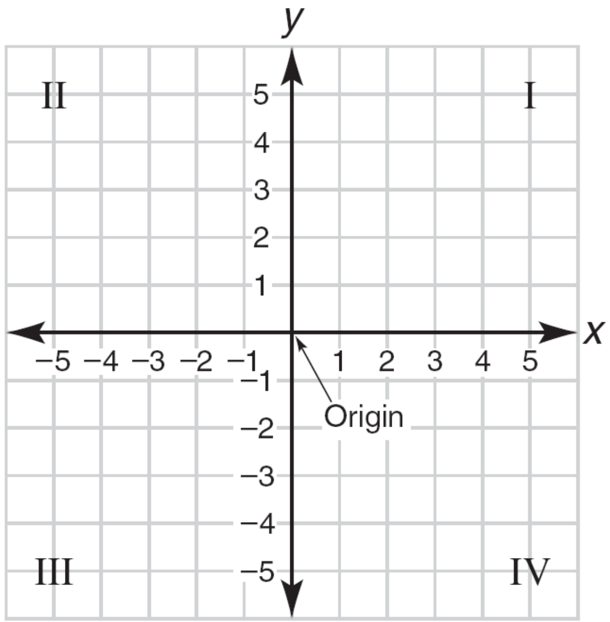

Figure 10.1

The Rectangular Coordinate System

The two axes separate the plane into four **quadrants** which are numbered counterclockwise I, II, III, and IV.

The point that marks the common origin of the two number lines is the **origin** of the rectangular coordinate system.

It is convenient to identify the origin as (0, 0); this notation indicates that the ***x* coordinate** (listed first) is 0 and also that the ***y* coordinate** (listed second) is 0.

The Rectangular Coordinate System

In the coordinate system, each point has the order (x, y) and is called an **ordered pair**. In Figure 10.2, the point (3, –2) for which x = 3 and y = –2 is shown.

The point (3, –2) is located by moving 3 units to the right of the origin and then 2 units down from the x axis.

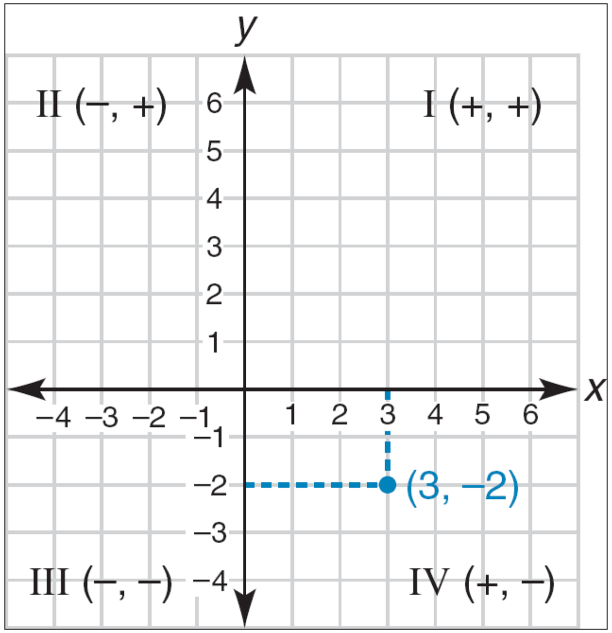

Figure 10.2

The Rectangular Coordinate System

The dashed lines shown emphasize the reason why the grid is called the *rectangular coordinate system*.

The point (3, –2) is located in Quadrant IV. In Figure 10.2, ordered pairs of positive and negative signs characterize the signs of the coordinates of a point located in each quadrant.

Example 1

Plot points *A* (–3, 4) and *B* (2, 4), and find the distance between them.

Solution:

Point *A* is located by moving 3 units to the left of the origin and then 4 units up from the *x* axis.

Point *B* is located by moving 2 units to the right of the origin and then 4 units up from the *x* axis.

In Figure 10.3, $\overline{AB}$ is a horizontal segment.

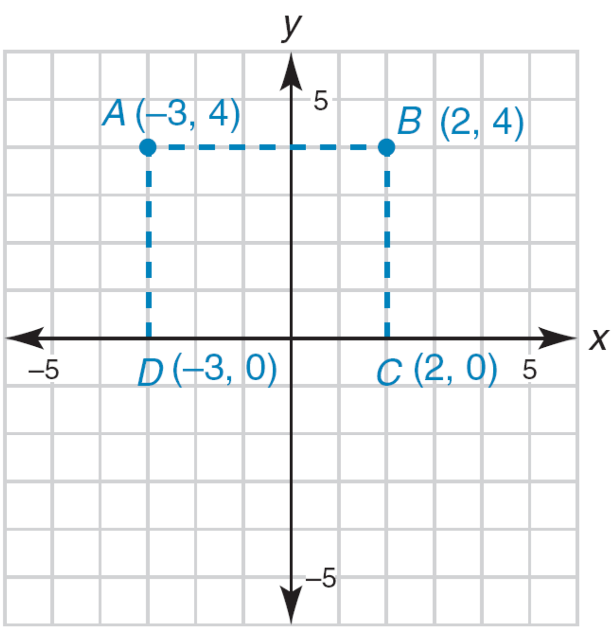

Figure 10.3

Example 1 – *Solution*

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In the rectangular coordinate system, *ABCD* is a rectangle in which *DC* = 5; $\overline{DC}$ is easily measured because it lies on the *x* axis.

Because the opposite sides of a rectangle are congruent, it follows that *AB* = 5.

The Rectangular Coordinate System

In Example 1, the points (–3, 4) and (2, 4) have the same y coordinates.

In this case, the distance between the points on a horizontal line is merely the positive difference in the x coordinates; thus, the distance between A and B is $2 – (–3)$, or 5.

It is also easy to find the distance between two points on a vertical line.

The Rectangular Coordinate System

When the *x* coordinates of two points are the same, the distance between points is the positive difference in the *y* coordinates.

In Figure 10.3, where *C* is (2, 0) and *B* is (2, 4), the distance between the points is 4 – 0 or 4.

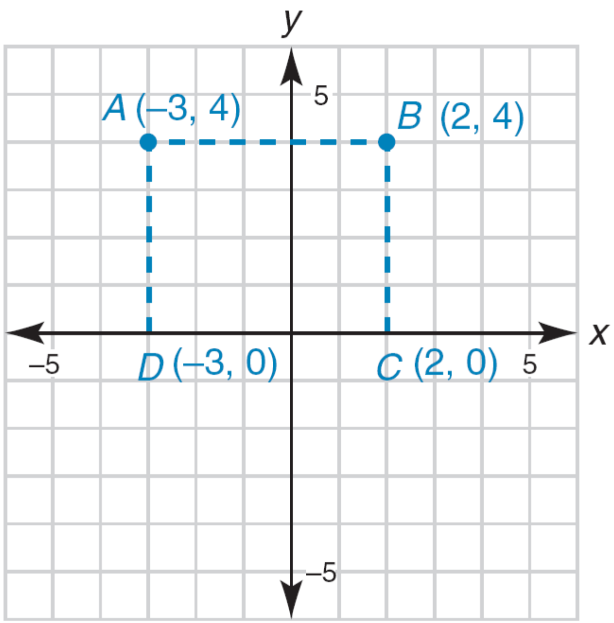

Figure 10.3

The Rectangular Coordinate System

In the following definition, the repeated y coordinates for the endpoints of the line segment indicate that the segment is horizontal.

Definition

Given points $A(x_1, y_1)$ and $B(x_2, y_1)$ on a horizontal line segment $\overline{AB}$, the distance between these points is

$$AB = x_2 - x_1 \text{ if } x_2 > x_1$$

or $$AB = x_1 - x_2 \text{ if } x_1 > x_2$$

The Rectangular Coordinate System

In the following definition, repeated *x* coordinates for the endpoints of a line segment determine a vertical line segment.

In both definition, the distance is found by subtracting the smaller from the larger of the two unequal coordinates.

Definition

Given points $C(x_1, y_1)$ and $D(x_1, y_2)$ on a vertical line segment $\overline{CD}$, the distance between these points is

$$CD = y_2 - y_1 \text{ if } y_2 > y_1$$

or $$CD = y_1 - y_2 \text{ if } y_1 > y_2$$

THE DISTANCE FORMULA

The Distance Formula

The following formula enables us to find the distance between two points that lie on a "slanted" line.

Theorem 10.1.1 (Distance Formula)
The distance between two points (x_1, y_1) and (x_2, y_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 3

In Figure 10.6, find the distance between points $A(5, -1)$ and $B(-1, 7)$.

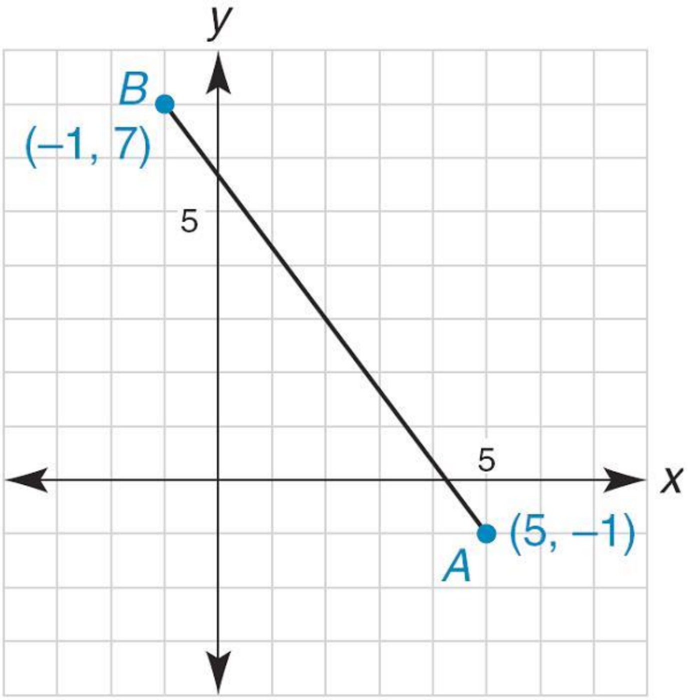

Figure 10.6

Example 3 – *Solution*

Using the Distance Formula and choosing $x_1 = 5$ and $y_1 = -1$ (from point *A*) and $x_2 = -1$ and $y_2 = 7$ (from point *B*), we obtain

$$
\begin{aligned}
d &= \sqrt{(-1 - 5)^2 + [7 - (-1)]^2} \\
&= \sqrt{(-6)^2 + (8)^2} \\
&= \sqrt{36 + 64} \\
&= \sqrt{100} \\
&= 10
\end{aligned}
$$

THE MIDPOINT FORMULA

The Midpoint Formula

In Figure 10.8, point M is the midpoint of $\overline{AB}$. It will be shown in Example 5(a) that M is the point (2, 3).

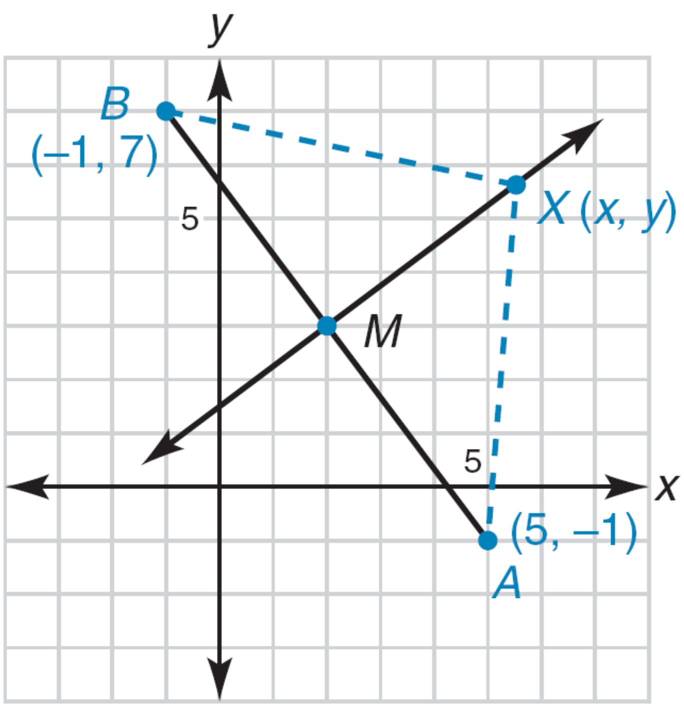

Figure 10.8

The Midpoint Formula

A generalized midpoint formula is given in Theorem 10.1.2. The result shows that the coordinates of the midpoint *M* of a line segment are the averages of the coordinates of the endpoints.

Theorem 10.1.2 (Midpoint Formula)

The midpoint *M* of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ has coordinates x_M and y_M, where

$$(x_M, y_M) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

that is,

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example 5

Use the Midpoint Formula to find the midpoint of the line segment joining:

a) (5, –1) and (–1, 7) **b)** (a, b) and (c, d)

Solution:

a) Using the Midpoint Formula and setting $x_1 = 5$, $y_1 = -1$, $x_2 = -1$, and $y_2 = 7$, we have

$$M = \left(\frac{5 + (-1)}{2}, \frac{-1 + 7}{2}\right)$$

$$= \left(\frac{4}{2}, \frac{6}{2}\right)$$

Example 5 – *Solution*

cont'd

So $\qquad M = (2, 3)$

b) Using the Midpoint Formula and setting $x_1 = a$, $y_1 = b$, $x_2 = c$, and $y_2 = d$, we have

$$M = \left(\frac{a + c}{2}, \frac{b + d}{2}\right)$$