Chapter 1

Line and Angle Relationships

1.4 Angles and Their Relationships

Angles and Their Relationships

Definition

An **angle** is the union of two rays that share a common endpoint.

In Figure 1.46, in which $\overrightarrow{BA}$ and $\overrightarrow{BC}$ have the common endpoint *B.* As shown, this angle is represented by $\angle ABC$ or $\angle CBA$.

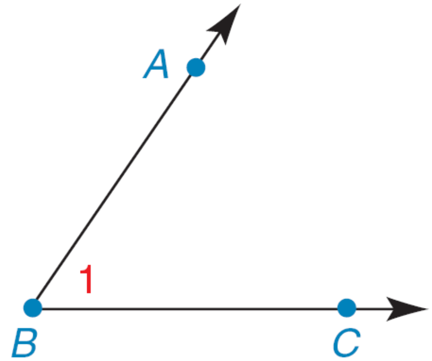


Figure 1.46

The rays *BA and BC* are known as the **sides** of the angle. *B*, the common endpoint of these rays, is known as the **vertex** of the angle. When three letters are used to name an angle, the vertex is always named in the middle.

Angles and Their Relationships

Recall that a single letter or numeral may be used to name the angle. The angle in Figure 1.46 may be described as $\angle B$ (the vertex of the angle) or as $\angle 1$.

In set notation, we see that $\angle B = \overrightarrow{BA} \cup \overrightarrow{BC}$

Postulate 8 (Protractor Postulate)

The measure of an angle is a unique positive number.

TYPES OF ANGLES

Types of Angles

An angle whose measure is less than 90° is an **acute angle.**

If the angle’s measure is exactly 90°, the angle is a **right angle.**

If the angle’s measure is between 90° and 180°, the angle is **obtuse.**

An angle whose measure is exactly 180° is a **straight angle;** alternatively, a straight angle is one whose sides form opposite rays (a straight line).

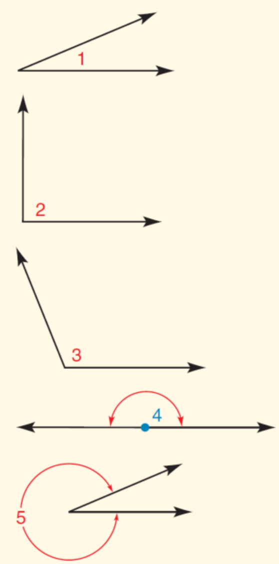
Types of Angles

A **reflex angle** is one whose measure is between 180° and 360°. See Table 1.4

TABLE 1.4

Angles

Angle	*Example*
Acute (1)	$m\angle 1 = 23°$
Right (2)	$m\angle 2 = 90°$
Obtuse (3)	$m\angle 3 = 112°$
Straight (4)	$m\angle 4 = 180°$
Reflex (5)	$m\angle 5 = 337°$



NOTE: An arc is necessary in indicating a reflex angle, and it can be used to indicate a straight angle as well.

Types of Angles

In Figure 1.47, $\angle ABC$ contains the noncollinear points A, B, and C. Unless otherwise stated or indicated (by an arc), $\angle ABC$ is an acute angle.

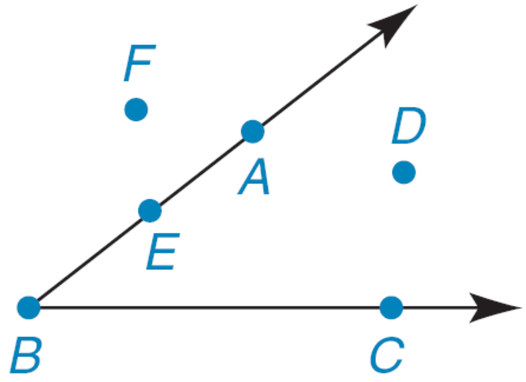


Figure 1.47

The three points A, B, and C also determine a plane.

Types of Angles

The plane containing $\angle ABC$ is separated into three subsets by the angle:

Points such as D are said to be in the *interior* of $\angle ABC$.

Points such as E are said to be *on* $\angle ABC$.

Points such as F are said to be in the *exterior* of $\angle ABC$.

Types of Angles

With this description, it is possible to state the Angle-Addition Postulate, which is the counterpart of the Segment-Addition Postulate! Consider Figure 1.48 as you read Postulate 9.

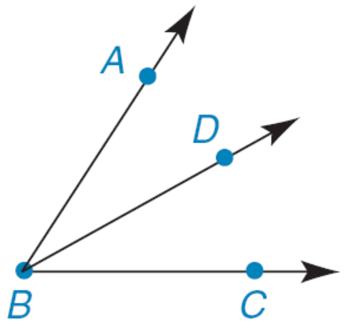


Figure 1.48

Postulate 9 (Angle-Addition Postulate)

If a point *D* lies in the interior of an angle *ABC*, then $m\angle ABD + m\angle DBC = m\angle ABC$.

Example 1

Use Figure 1.48 to find m$\angle$*ABC* if:

a) m$\angle$ *ABD* = 27° and m $\angle$ *DBC* = 42°

b) m$\angle$ *ABD* = *x*° and m $\angle$ *DBC* = (2*x* – 3)°

Solution:

a) Using the Angle-Addition Postulate,

$$m\angle ABC = m\angle ABD + m\angle DBC.$$

That is, m$\angle$ *ABC* = 27° + 42° = 69°.

b) $m\angle ABC = m\angle ABD + m\angle DBC$

$$= x° + (2x - 3)°$$

$$= (3x - 3)°$$

CLASSIFYING PAIRS OF ANGLES

Classifying Pairs of Angles

Many angle relationships involve exactly two angles (a pair)—never more than two angles and never less than two angles!

In Figure 1.48, $\angle ABD$ and $\angle DBC$ are said to be *adjacent* angles.

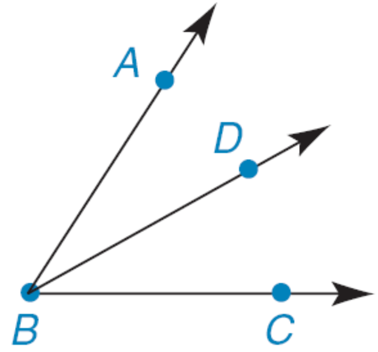


Figure 1.48

Classifying Pairs of Angles

In this description, the term *adjacent* means that angles lie “next to” each other; in everyday life, one might say that the Subway sandwich shop is adjacent to the Baskin-Robbins ice cream shop.

When two angles are adjacent, they have a common vertex and a common side between them.

In Figure 1.48, $\angle ABC$ and $\angle ABD$ are not adjacent because they have interior points in common.

Classifying Pairs of Angles

Definition

Two angles are **adjacent** (adj. ∠s) if they have a common vertex and a common side between them.

Definition

Congruent angles (**≅∠s**) are two angles with the same measure.

Congruent angles must coincide when one is placed over the other. (Do not consider that the sides appear to have different lengths; remember that rays are infinite in length!)

Classifying Pairs of Angles

In symbols, $\angle 1 \cong \angle 2$ if $m\angle 1 = m\angle 2$. In Figure 1.49, similar markings (arcs) indicate that two angles are congruent; thus, $\angle 1 \cong \angle 2$.

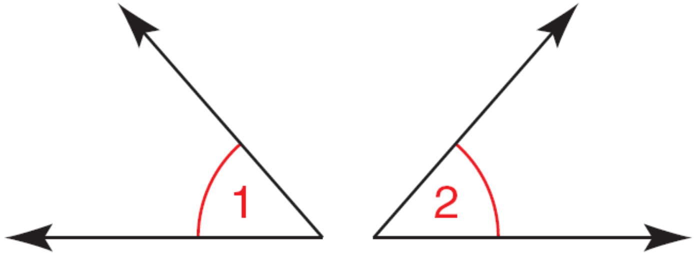


Figure 1.49

Example 2

Given: $\angle 1 \cong \angle 2$

$m\angle 1 = 2x + 15$

$m\angle 2 = 3x - 2$

Find: x

Solution:

$\angle 1 \cong \angle 2$ means $m\angle 1 = m\angle 2$.

Therefore,

$$2x + 15 = 3x - 2$$

$$17 = x \quad \text{or} \quad x = 17$$

Note: $m\angle 1 = 2(17) + 15 = 49°$ and $m\angle 2 = 3(17) - 2 = 49°$.

Classifying Pairs of Angles

Definition

The **bisector** of an angle is the ray that separates the given angle into two congruent angles.

With *P* in the interior of $\angle MNQ$ so that $\angle MNP \cong \angle PNQ$, $\overrightarrow{NP}$ is said to **bisect** $\angle MNQ$.

Equivalently, $\overrightarrow{NP}$ is the bisector or angle-bisector of $\angle MNQ$.

Classifying Pairs of Angles

On the basis of Figure 1.50, possible consequences of the definition of bisector of an angle are

$m\angle MNP = m\angle PNQ$

$m\angle MNQ = 2(m\angle PNQ)$

$m\angle MNQ = 2(m\angle MNP)$

$m\angle PNQ = \frac{1}{2}(m\angle MNQ)$

$m\angle MNP = \frac{1}{2}(m\angle MNQ)$

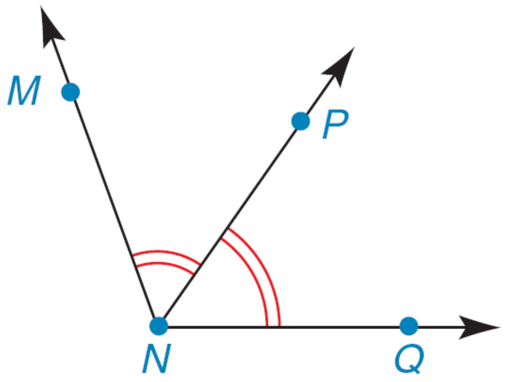


Figure 1.50

Classifying Pairs of Angles

Definition

Two angles are **complementary** if the sum of their measures is 90°. Each angle in the pair is known as the **complement** of the other angle.

Angles with measures of 37° and 53° are complementary. The 37° angle is the complement of the 53° angle, and vice versa.

If the measures of two angles are x and y and it is known that $x + y = 90°$, then these two angles are complementary.

Classifying Pairs of Angles

Definition

Two angles are **supplementary** if the sum of their measures is 180°. Each angle in the pair is known as the **supplement** of the other angle.

Example 3

Given that m$\angle 1 = 29°$, find:

a) the complement x of $\angle 1$ **b)** the supplement y of $\angle 1$

Solution:

a) $x + 29 = 90$,

so $x = 61°$; complement = 61°

b) $y + 29 = 180$,

so $y = 151°$; supplement = 151°

Classifying Pairs of Angles

When two straight lines intersect, the pairs of nonadjacent angles in opposite positions are known as **vertical angles.**

In Figure 1.51, $\angle 5$ and $\angle 6$ are vertical angles (as are $\angle 7$ and $\angle 8$).

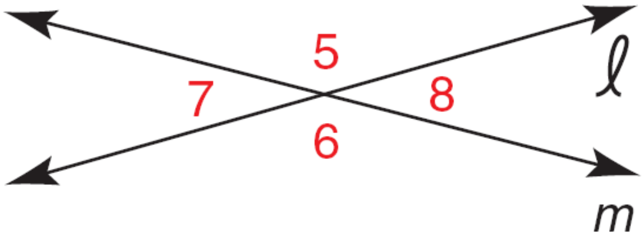


Figure 1.51

In addition, $\angle 5$ and $\angle 7$ can be described as adjacent and supplementary angles, as can $\angle 5$ and $\angle 8$.

If $m\angle 7 = 30°$, what is $m\angle 5$ and what is $m\angle 8$? It is true in general that vertical angles are congruent.

Classifying Pairs of Angles

Recall the Addition and Subtraction Properties of Equality:

If $a = b$ and $c = d$, then $a \pm c = b \pm d$.

These principles can be used in solving a system of equations, such as the following:

$$
\begin{array}{rl}
x + y &= 5 \\
2x - y &= 7 \\
\hline
3x \quad\quad &= 12 \\
x &= 4
\end{array}
$$

(left and right sides are added)

Classifying Pairs of Angles

We can substitute 4 for x in either equation to solve for y:

$$x + y = 5$$

$$4 + y = 5 \qquad \text{(by substitution)}$$

$$y = 1$$

If $x = 4$ and $y = 1$, then $x + y = 5$ and $2x - y = 7$.

When each term in an equation is multiplied by the same nonzero number, the solutions of the equation are not changed.

Classifying Pairs of Angles

For instance, the equations $2x – 3 = 7$ and $6x – 9 = 21$ (each term multiplied by 3) both have the solution $x = 5$.

Likewise, the values of x and y that make the equation $4x + y = 180$ true also make the equation $16x + 4y = 720$ (each term multiplied by 4) true.

CONSTRUCTIONS WITH ANGLES

Constructions with angles

Construction 3

To construct an angle congruent to a given angle.

Given: $\angle RST$ in Figure 1.52(a)

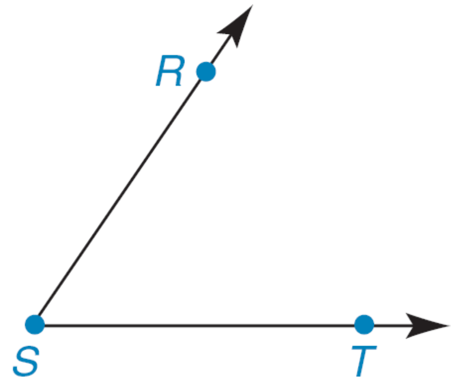


Figure 1.52(a)

Construct: With $\overrightarrow{PQ}$ as one side, $\angle NPQ \cong \angle RST$

Constructions with angles

Construction:
Figure 1.52(b): With a compass, mark an arc to intersect both sides of $\angle RST$ at points G and H.

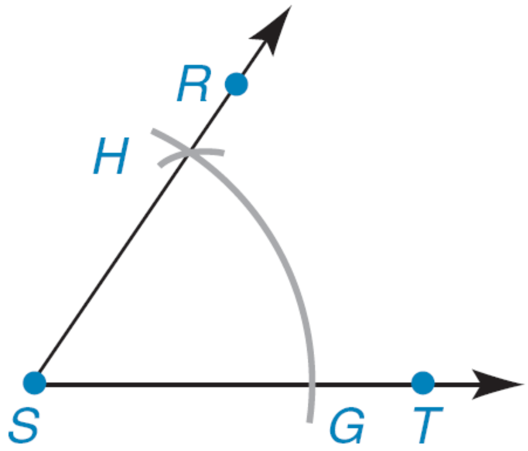


Figure 1.52(b)

Constructions with angles

Figure 1.52(c): Without changing the radius, mark an arc to intersect $\overrightarrow{PQ}$ at *K* and the “would-be” second side of $\angle NPQ$.

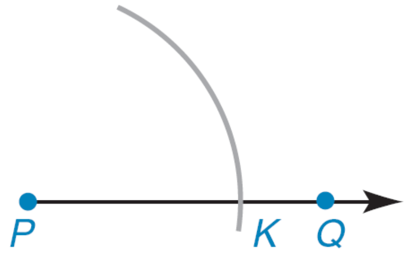


Figure 1.52(c)

Figure 1.52(b): Now mark an arc to measure the distance from *G* to *H*.

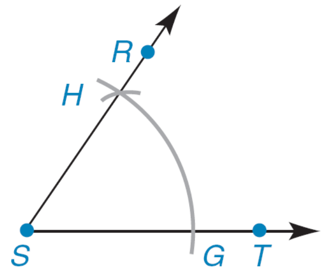


Figure 1.52(b)

Constructions with angles

Figure 1.52(d): Using the same radius, mark an arc with *K* as center to intersect the would-be second side of the desired angle. Now draw the ray from *P* through the point of intersection of the two arcs.

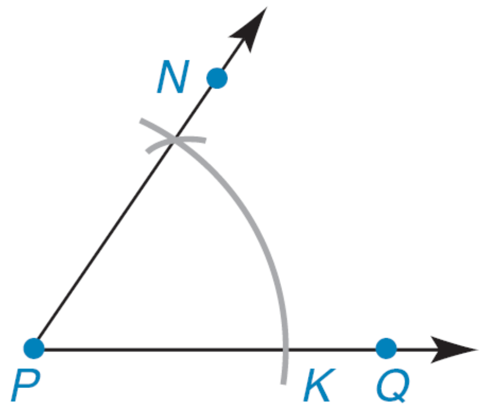


Figure 1.52(d)

Just as a line segment can be bisected, so can an angle. This takes us to a fourth construction method.

Constructions with angles

Construction 4

To construct the angle bisector of a given angle.

Given: $\angle PRT$ in Figure 1.53(a)

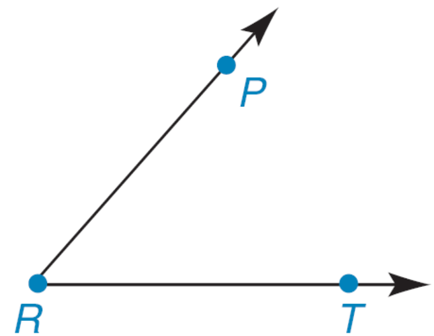


Figure 1.53(a)

Construct: $\overrightarrow{RS}$ so that $\angle PRS \cong \angle SRT$

Constructions with angles

Construction:

Figure 1.53(b): Using a compass, mark an arc to intersect the sides of $\angle PRT$ at points *M* and *N*.

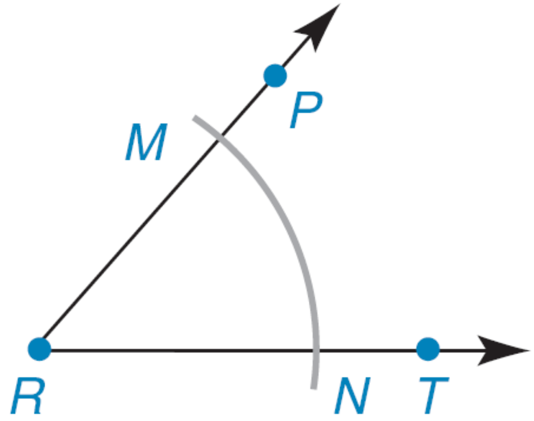


Figure 1.53(b)

Constructions with angles

Figure 1.53(c): Now, with *M* and *N* as centers, mark off two arcs with equal radii to intersect at point *S* in the interior of $\angle PRT$, as shown.

Now draw ray *RS*, the desired angle bisector.

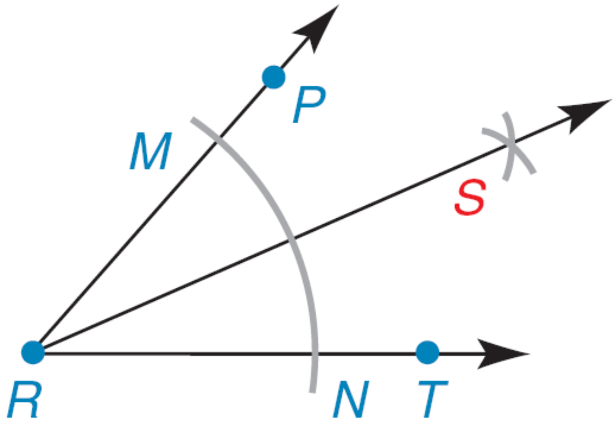


Figure 1.53(c)

Constructions with angles

Reasoning from the definition of an angle bisector, the Angle-Addition Postulate, and the Protractor Postulate, we can justify the following theorem.

Theorem 1.4.1

There is one and only one bisector for a given angle.

This theorem is often stated, “The bisector of an angle is unique.”