Chapter 1 Line and Angle Relationships

1.3 Early Definitions and Postulates

A MATHEMATICAL SYSTEM

A Mathematical System

Like algebra, the branch of mathematics called geometry is a **mathematical system.**

The formal study of a mathematical system begins with undefined terms.

Building on this foundation, we can then define additional terms. Once the terminology is sufficiently developed, certain properties (characteristics) of the system become apparent.

These properties are known as **axioms** or **postulates** of the system; more generally, such statements are called **assumptions.**

A Mathematical System

Once we have developed a vocabulary and accepted certain postulates, many principles follow logically as we apply deductive methods. These statements can be proved and are called **theorems.**

The following box summarizes the components of a mathematical system (sometimes called a logical system or deductive system).

FOUR PARTS OF A MATHEMATICAL SYSTEM	
1. Undefined terms **2.** Defined terms	vocabulary
3. Axioms or postulates **4.** Theorems	principles

CHARACTERISTICS OF A GOOD DEFINITION

Characteristics of a Good Definition

Terms such as *point, line,* and *plane* are classified as undefined because they do not fit into any set or category that has been previously determined.

Terms that *are* defined, however, should be described precisely.

But what is a good definition? A good definition is like a mathematical equation written using words.

A good definition must possess four characteristics.

Characteristics of a Good Definition

Definition

An **isosceles triangle** is a triangle that has two congruent sides.

In the definition, notice that:

(1) The term being defined—*isosceles triangle*—is named.

(2) The term being defined is placed into a larger category (a type of *triangle*).

(3) The distinguishing quality (that two sides of the triangle are congruent) is included.

Characteristics of a Good Definition

(4) The *reversibility* of the definition is illustrated by these statements:

"If a triangle is isosceles, then it has two congruent sides."

"If a triangle has two congruent sides, then it is an isosceles triangle."

CHARACTERISTICS OF A GOOD DEFINITION

1. It names the term being defined.
2. It places the term into a set or category.
3. It distinguishes the defined term from other terms without providing unnecessary facts.
4. It is reversible.

Characteristics of a Good Definition

The reversibility of a definition is achieved by using the phrase “if and only if.” For instance, we could define *congruent angles* by saying “Two angles are congruent if and only if these angles have equal measures.”

The “if and only if” statement has the following dual meaning:

“If two angles are congruent, then they have equal measures.”

“If two angles have equal measures, then they are congruent.”

Characteristics of a Good Definition

When represented by a Venn Diagram,
the definition above would relate set
C = {congruent angles} to set
E = {angles with equal measures} as shown
in Figure 1.30.

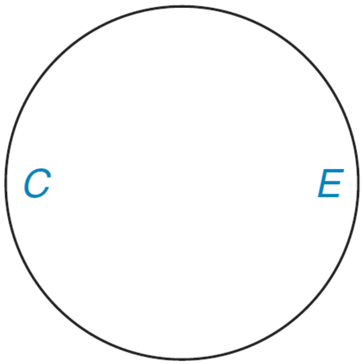

Figure 1.30

The sets C and E are identical and are known as **equivalent sets**.

Definition

A **line segment** is the part of a line that consists of two points, known as endpoints, and all points between them.

Example 1

State the four characteristics of a good definition of the term “line segment.”

1. The term being defined, *line segment*, is clearly present in the definition.

2. A line segment is defined as part of a line (a category).

3. The definition distinguishes the line segment as a specific part of a line.

Example 1

cont'd

4. The definition is reversible.

i) A line segment is the part of a line between and including two points.

ii) The part of a line between and including two points is a line segment.

INITIAL POSTULATES

Initial Postulates

Recall that a postulate is a statement that is assumed to be true.

Postulate 1

Through two distinct points, there is exactly one line.

Postulate 1 is sometimes stated in the form "Two points determine a line." See Figure 1.32, in which points C and D determine exactly one line, namely, $\overleftrightarrow{CD}$.

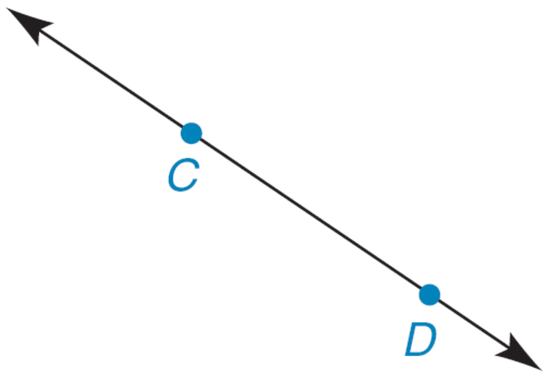

Figure 1.32

Initial Postulates

Of course, Postulate 1 also implies that there is a unique line segment determined by two distinct points used as endpoints.

In Figure 1.31, points *A* and *B* determine $\overline{AB}$.

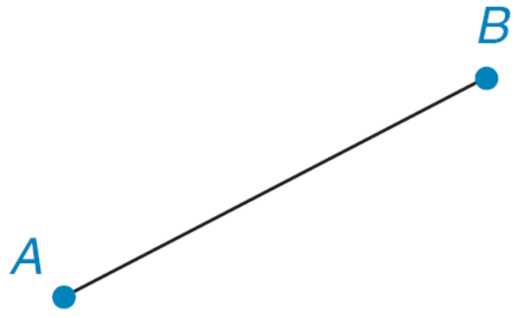

Figure 1.31

Example 2

In Figure 1.33, how many distinct lines can be drawn through

a) point *A*?

b) both points *A* and *B* at the same time?

c) all points *A*, *B*, and *C* at the same time?

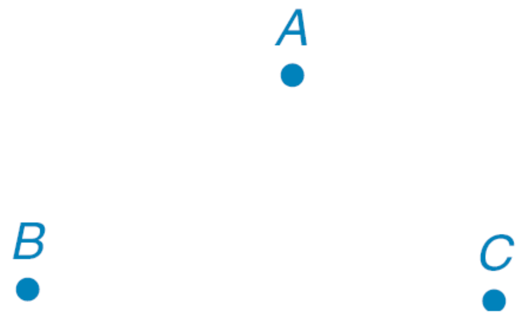

Figure 1.33

Solution:

a) An infinite (countless) number

b) Exactly one

c) No line contains all three points.

Initial Postulates

The symbol for line segment *AB*, named by its endpoints, is $\overline{AB}$.

Omission of the bar from $\overline{AB}$, as in *AB*, means that we are considering the *length* of the segment. These symbols are summarized in Table 1.3.

TABLE 1.3

Symbol	Words for Symbol	Geometric Figure
$\overleftrightarrow{AB}$	Line AB	
$\overline{AB}$	Line segment AB	
AB	Length of segment AB	A number

Initial Postulates

A ruler is used to measure the length of a line segment such as $\overline{AB}$.

This length may be represented by *AB* or *BA* (the order of *A* and *B* is not important). However, *AB* must be a positive number.

Initial Postulates

Postulate 2 (Ruler Postulate)

The measure of any line segment is a unique positive number.

We wish to call attention to the term *unique* and to the general notion of uniqueness.

The Ruler Postulate implies the following:

1. There exists a number measure for each line segment.

2. Only *one* measure is permissible.

Initial Postulates

Characteristics 1 and 2 are both necessary for uniqueness!

Other phrases that may replace the term *unique* include

One and only one

Exactly one

One and no more than one

A more accurate claim than the commonly heard statement "The shortest distance between two points is a straight line" is found in the following definition.

Initial Postulates

Definition

The **distance** between two points *A* and *B* is the length of the line segment $\overline{AB}$ that joins the two points.

Postulate 3 (Segment-Addition Postulate)

If *X* is a point of $\overline{AB}$ and *A*-*X*-*B*, then $AX + XB = AB$.

Initial Postulates

Definition

Congruent (≅) line **segments** are two line segments that have the same length.

In general, geometric figures that can be made to coincide (fit perfectly one on top of the other) are said to be **congruent.**

The symbol ≅ is a combination of the symbol ~, which means that the figures have the same shape, and =, which means that the corresponding parts of the figures have the same measure.

Initial Postulates

In Figure 1.35, $\overline{AB} \cong \overline{CD}$, but $\overline{AB} \not\cong \overline{EF}$ (meaning that $\overline{AB}$ and $\overline{EF}$ are not congruent). Does it appear that $\overline{CD} \cong \overline{EF}$?

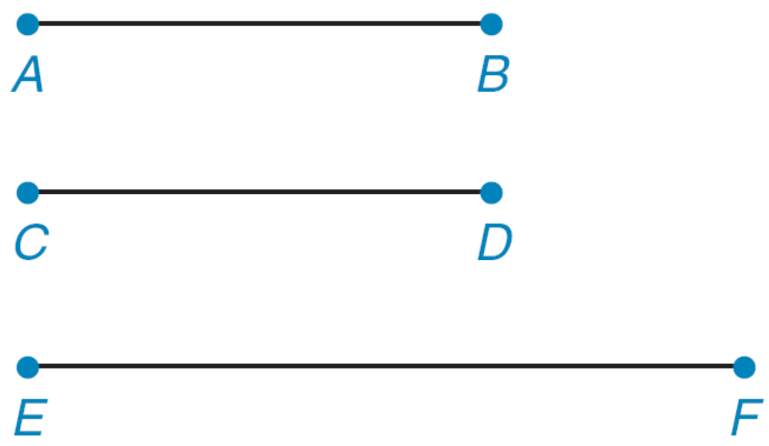

Figure 1.35

Initial Postulates

Definition

The **midpoint** of a line segment is the point that separates the line segment into two congruent parts.

In Figure 1.36, if *A*, *M*, and *B* are collinear and $\overline{AM} \cong \overline{MB}$, then *M* is the **midpoint** of $\overline{AB}$. Equivalently, *M* is the midpoint of $\overline{AB}$ if *AM* = *MB*. Also, if $\overline{AM} \cong \overline{MB}$, then $\overline{CD}$ is described as a **bisector** of $\overline{AB}$.

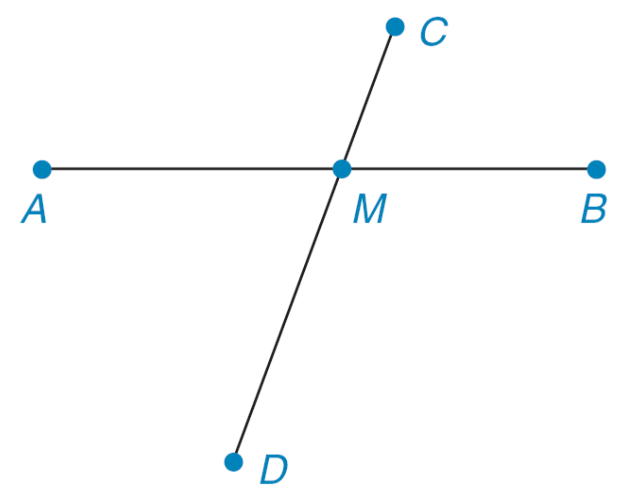

Figure 1.36

Initial Postulates

If M is the midpoint of $\overline{AB}$ in Figure 1.36, we can draw any of these conclusions:

$$AM = MB \qquad MB = \frac{1}{2}(AB) \qquad AB = 2(MB)$$

$$AM = \frac{1}{2}(AB) \qquad AB = 2(AM)$$

Initial Postulates

Definition

Ray *AB*, denoted by $\overrightarrow{AB}$, is the union of $\overline{AB}$ and all points *X* on $\overleftrightarrow{AB}$ such that *B* is between *A* and *X*.

In Figure 1.37, $\overleftrightarrow{AB}$, $\overrightarrow{AB}$, and $\overrightarrow{BA}$ are shown in that order; note that $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are not the same ray.

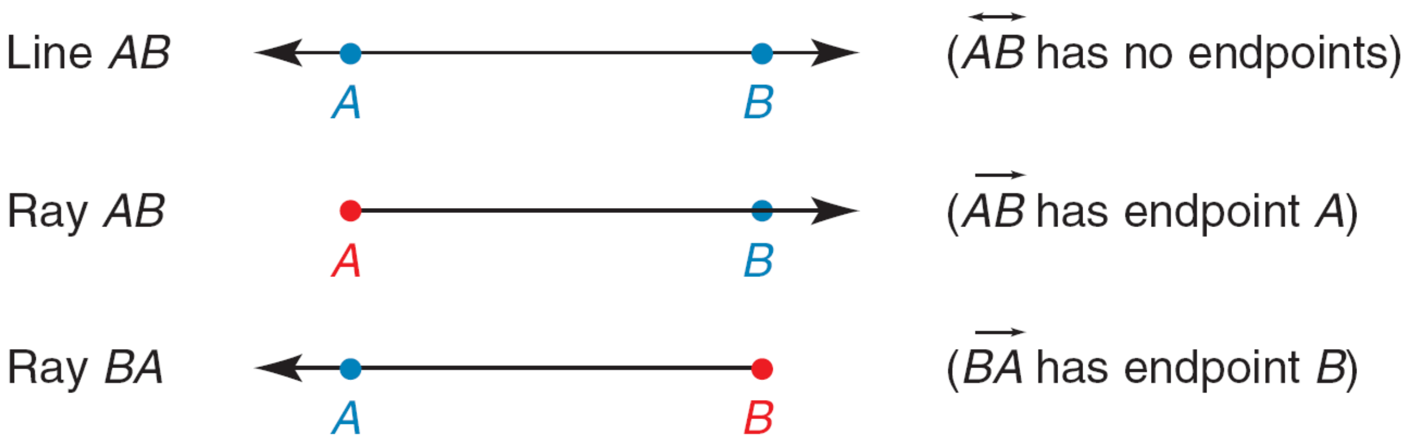

Figure 1.37

Initial Postulates

Opposite rays are two rays with a common endpoint; also, the union of opposite rays is a straight line.

In Figure 1.39(a), $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are opposite rays.

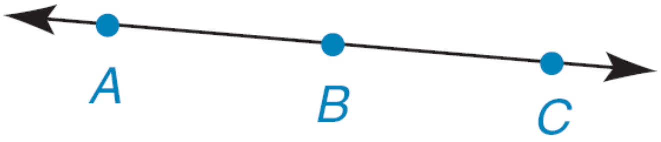

Figure 1.39(a)

Initial Postulates

The **intersection** of two geometric figures is the set of points that the two figures have in common.

In everyday life, the intersection of Bradley Avenue and Neil Street is the part of the roadway that the two roads have in common (Figure 1.38).

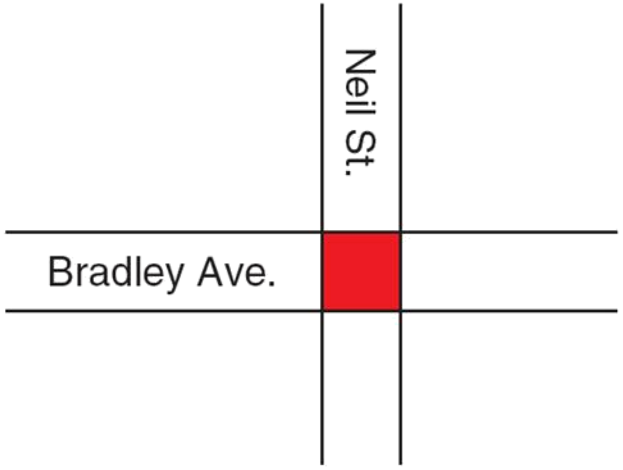

Figure 1.38

Initial Postulates

Postulate 4

If two lines intersect, they intersect at a point.

When two lines share two (or more) points, the lines coincide; in this situation, we say there is only one line.

In Figure 1.39(a), $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ are the same as $\overleftrightarrow{AC}$. In Figure 1.39(b), lines ℓ and m intersect at point P.

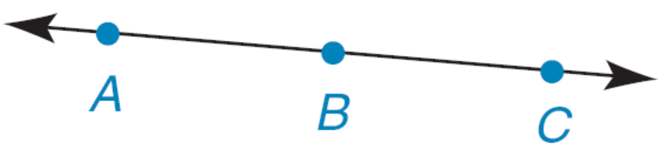

Figure 1.39(a)

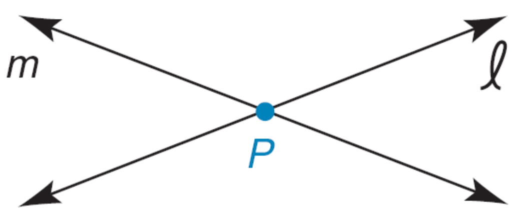

Figure 1.39(b)

Initial Postulates

Definition

Parallel lines are lines that lie in the same plane but do not intersect.

Another undefined term in geometry is **plane.** A plane is two-dimensional; that is, it has infinite length and infinite width but no thickness.

Except for its limited size, a flat surface such as the top of a table could be used as an example of a plane.

Initial Postulates

An uppercase letter can be used to name a plane. Because a plane (like a line) is infinite, we can show only a portion of the plane or planes, as in Figure 1.41.

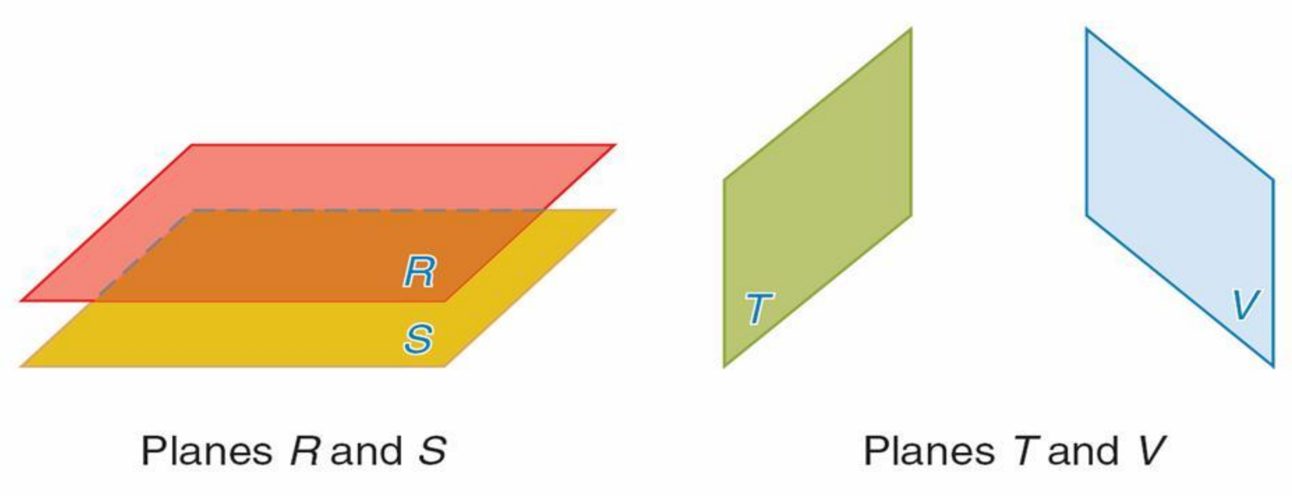

Figure 1.41

Initial Postulates

A plane is two-dimensional, consists of an infinite number of points, and contains an infinite number of lines.

Two distinct points may determine (or “fix”) a line; likewise, exactly three noncollinear points determine a plane.

Just as collinear points lie on the same line, **coplanar points** lie in the same plane.

Initial Postulates

In Figure 1.42, points *B*, *C*, *D*, and *E* are coplanar, whereas *A*, *B*, *C*, and *D* are noncoplanar.

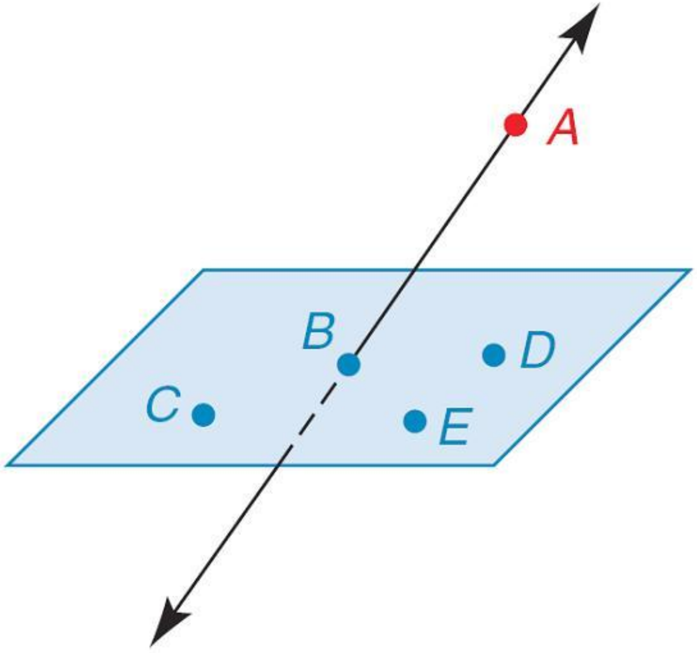

Figure 1.42

Initial Postulates

Points shown in figures are generally assumed to be coplanar unless otherwise stated. For instance, points *A*, *B*, *C*, *D*, and *E* are coplanar in Figure 1.43(a), as are points *F*, *G*, *H*, *J*, and *K* in Figure 1.43(b).

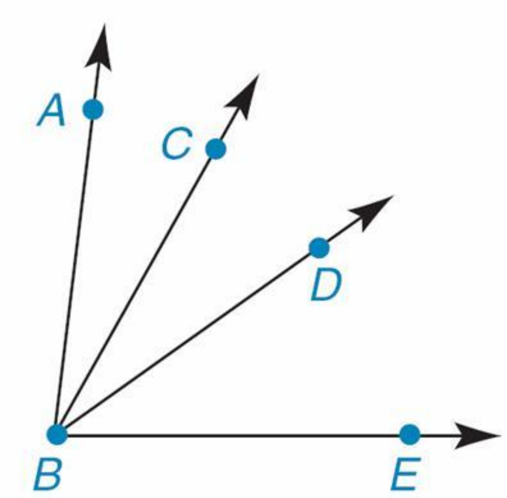

Figure 1.43(a)

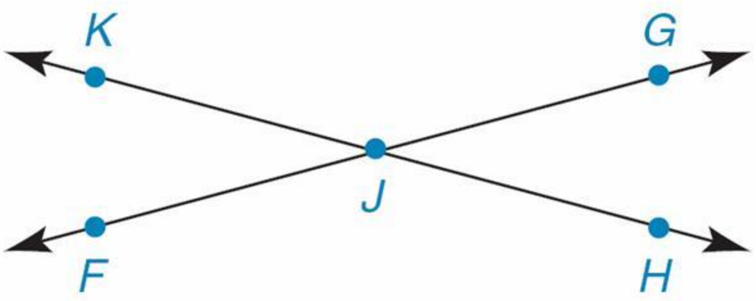

Figure 1.43(b)

Initial Postulates

Postulate 5

Through three noncollinear points, there is exactly one plane.

On the basis of Postulate 5, we can see why a three-legged table sits evenly but a four-legged table would "wobble" if the legs were of unequal length.

Space is the set of all possible points. It is three-dimensional, having qualities of length, width, and depth. When two planes intersect in space, their intersection is a line.

Initial Postulates

An opened greeting card suggests this relationship, as does Figure 1.44(a). This notion gives rise to our next postulate.

Postulate 6

If two distinct planes intersect, then their intersection is a line.

The intersection of two planes is infinite because it is a line. See Figure 1.44(a).

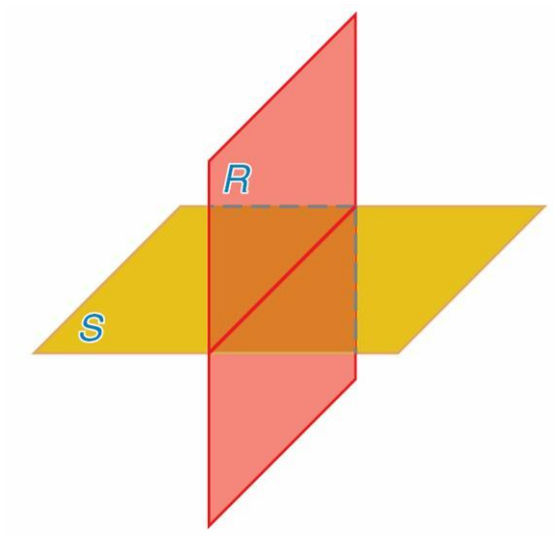

Figure 1.44(a)

Initial Postulates

If two planes do not intersect, then they are **parallel.**
The parallel **vertical** planes *R* and *S* in Figure 1.44(b) may remind you of the opposite walls of your classroom.

The parallel **horizontal** planes *M* and *N* in Figure 1.44(c) suggest the relationship between ceiling and floor.

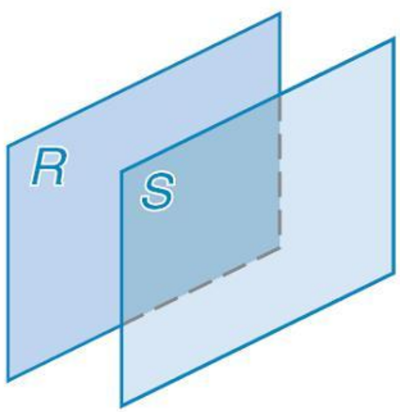

Figure 1.44(b)

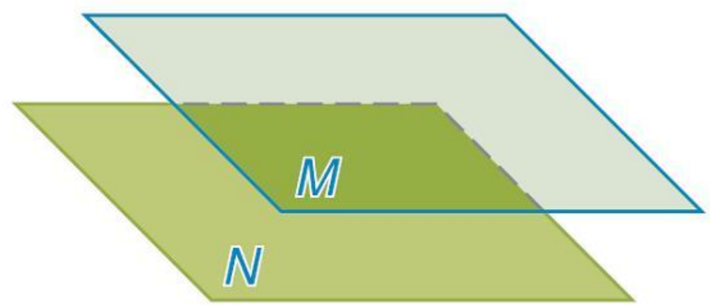

Figure 1.44(c)

Initial Postulates

Imagine a plane and two points of that plane, say points *A* and *B*. Now think of the line containing the two points and the relationship of $\overleftrightarrow{AB}$ to the plane. Perhaps your conclusion can be summed up as follows.

Postulate 7

Given two distinct points in a plane, the line containing these points also lies in the plane.

Initial Postulates

Theorem 1.3.1

The midpoint of a line segment is unique.

If *M* is the midpoint of $\overline{AB}$ in Figure 1.45, then no other point can separate $\overline{AB}$ into two congruent parts.

M is *the* point that is located $\frac{1}{2}$(*AB*) units from *A* (and from *B*).

Figure 1.45