Chapter 1

Line and Angle Relationships

1.1 Sets, Statements, and Reasoning

SETS

Sets

A **set** is any collection of objects, all of which are known as the *elements* of the set.

The statement $A = \{1, 2, 3\}$ is read, "*A* is the set of elements 1, 2, and 3." In geometry, geometric figures such as lines and angles are actually sets of points.

Where $A = \{1, 2, 3\}$ and $B = \{\text{counting numbers}\}$, *A* is a *subset* of *B* because each element in *A* is also in *B*; in symbols, $A \subseteq B$.

STATEMENTS

Statements

Definition

A **statement** is a set of words and/or symbols that collectively make a claim that can be classified as true or false.

Example 1

Classify each of the following as a true statement, a false statement, or neither.

1. 4 + 3 = 7

2. An angle has two sides.
(See Figure 1.1.)

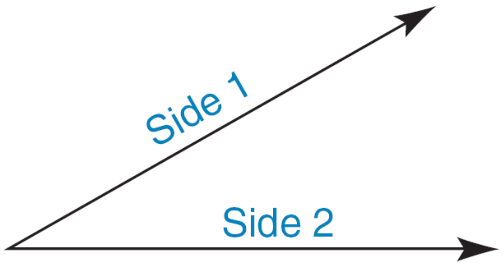


Figure 1.1

3. Robert E. Lee played shortstop for the Yankees.

4. 7 < 3 (This is read, "7 is less than 3.")

5. Look out!

Example 1 – *Solution*

1 and 2 are true statements; 3 and 4 are false statements; 5 is not a statement.

Some statements contain one or more *variables*; a **variable** is a letter that represents a number.

The claim "$x + 5 = 6$" is called an *open sentence* or *open statement* because it can be classified as true or false, depending on the replacement value of x.

Example 1 – *Solution*

cont'd

For instance, $x + 5 = 6$ is true if $x = 1$; for x not equal to 1, $x + 5 = 6$ is false.

Some statements containing variables are classified as true because they are true for all replacements.

Consider the Commutative Property of Addition, usually stated in the form $a + b = b + a$.

In words, this property states that the same result is obtained when two numbers are added in either order; for instance, when $a = 4$ and $b = 7$, it follows that $4 + 7 = 7 + 4$.

Example 1 – *Solution*

cont'd

The **negation** of a given statement *P* makes a claim opposite that of the original statement.

If the given statement is true, its negation is false, and vice versa. If *P* is a statement, we use ~*P* (which is read "not *P*") to indicate its negation.

Statements

A *compound* statement is formed by combining other statements used as “building blocks.”

In such cases, we may use letters such as P and Q to represent simple statements.

For example, the letter P may refer to the statement “$4 + 3 = 7$,” and the letter Q to the statement “Babe Ruth was a U.S. president.”

Statements

The statement “4 + 3 = 7 *and* Babe Ruth was a U.S. president” has the form *P and Q* and is known as the **conjunction** of *P* and *Q*.

The statement “4 + 3 = 7 *or* Babe Ruth was a U.S. president” has the form *P or Q* and is known as the **disjunction** of statement *P* and statement *Q*.

Statements

A conjunction is true only when *P* and *Q* are *both* true. A disjunction is false only when *P* and *Q* are *both* false.

See Tables 1.1 and 1.2.

TABLE 1.1
The Conjunction

P	Q	P and Q
T	T	T
T	F	F
F	T	F
F	F	F

TABLE 1.2
The Disjunction

P	Q	P or Q
T	T	T
T	F	T
F	T	T
F	F	F

Statements

The statement “If *P*, then *Q*,” known as a **conditional statement** (or **implication**), is classified as true or false as a whole.

A statement of this form can be written in equivalent forms; for instance, the conditional statement “If an angle is a right angle, then it measures 90 degrees” is equivalent to the statement “All right angles measure 90 degrees.”

Statements

In the conditional statement “If *P*, then *Q*,” *P* is the **hypothesis** and *Q* is the **conclusion**.

Hypothesis: Two sides of a triangle are equal in length.

Conclusion: Two angles of the triangle are equal in measure.

For the true statement “If *P*, then *Q*,” the hypothetical situation described in *P* implies the conclusion described in *Q*.

This type of statement is often used in reasoning, so we turn our attention to this matter.

REASONING

Reasoning

Success in the study of geometry requires vocabulary development, attention to detail and order, supporting claims, and thinking.

Reasoning is a process based on experience and principles that allows one to arrive at a conclusion.

The following types of reasoning are used to develop mathematical principles.

1.	Intuition	An inspiration leading to the statement of a theory
2.	Induction	An organized effort to test and validate the theory
3.	Deduction	A formal argument that proves the tested theory

Transformations

- **Intuition**

We are often inspired to think and say, “It occurs to me that. . . .” With **intuition**, a sudden insight allows one to make a statement without applying any formal reasoning.

When intuition is used, we sometimes err by “jumping” to conclusions. In a cartoon, the character having the “bright idea” (using intuition) is shown with a light bulb next to her or his head.

Example 5

Figure 1.3 is called a *regular pentagon* because its five sides have equal lengths and its five interior angles have equal measures. What do you suspect is true of the lengths of the dashed parts of lines from *B* to *E* and from *B* to *D*?

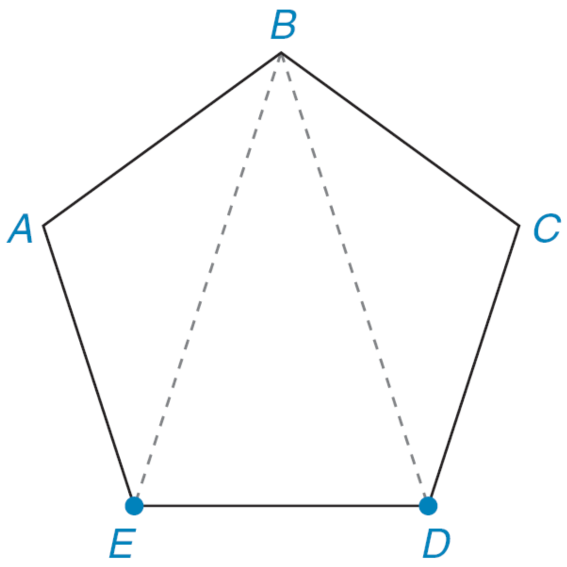


Figure 1.3

Example 5 – *Solution*

Intuition suggests that the lengths of the dashed parts of lines (known as *diagonals* of the pentagon) are the same.

Note 1: Using induction (and a *ruler*), we can verify that this claim is true.

Note 2: We could use deduction to prove that the two diagonals do indeed have the same length.

Transformations

The role intuition plays in formulating mathematical thoughts is truly significant. But to have an idea is not enough!

Testing a theory may lead to a revision of the theory or even to its total rejection. If a theory stands up to testing, it moves one step closer to becoming mathematical law.

Induction

We often use specific observations and experiments to draw a general conclusion. This type of reasoning is called **induction**.

Transformations

As you would expect, the observation/experimentation process is common in laboratory and clinical settings.

Chemists, physicists, doctors, psychologists, weather forecasters, and many others use collected data as a basis for drawing conclusions.

Example 7

In a geometry class, you have been asked to measure the three interior angles of each triangle in Figure 1.4. You discover that triangles I, II, and IV have two angles (as marked) that have equal measures. What may you conclude?

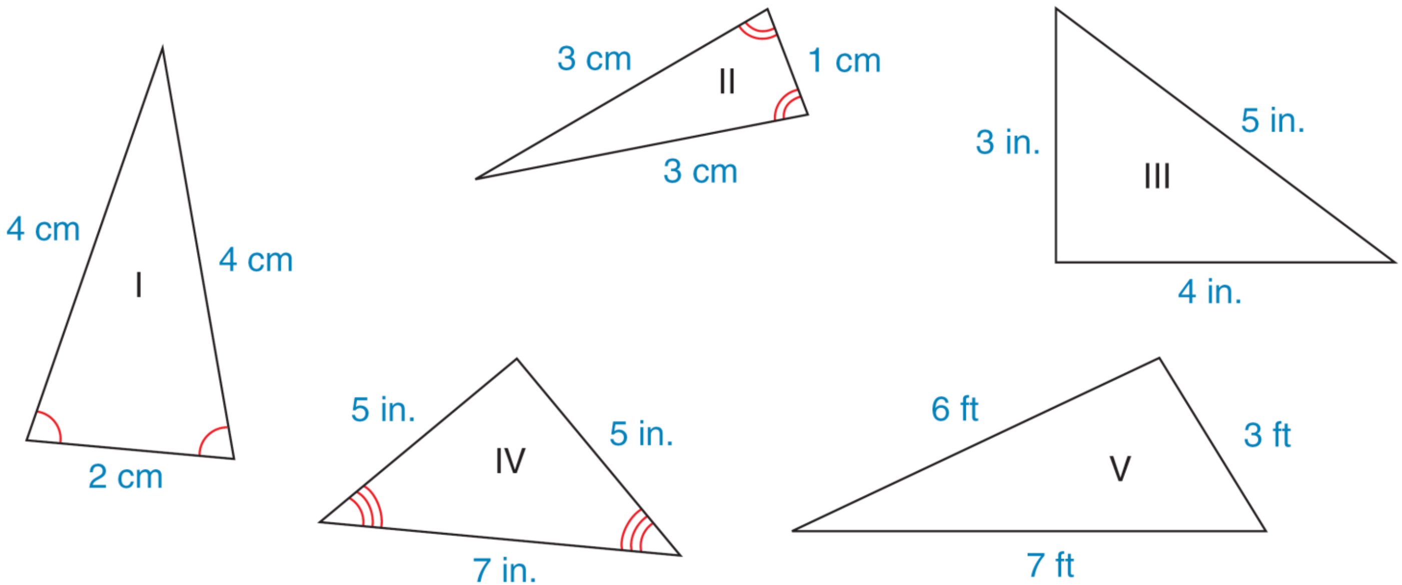


Figure 1.4

Example 7

Conclusion

The triangles that have two sides of equal length also have two angles of equal measure.

Deduction

Definition

Deduction is the type of reasoning in which the knowledge and acceptance of selected assumptions guarantee the truth of a particular conclusion.

In Example 8, we will illustrate a **valid argument**, a form of deductive reasoning used frequently in the development of geometry.

Reasoning

In this form, at least two statements are treated as facts; these assumptions are called the *premises* of the argument.

On the basis of the premises, a particular *conclusion* must follow. This form of deduction is called the **Law of Detachment**.

Example 8

If you accept the following statements 1 and 2 as true, what must you conclude?

1. If a student plays on the Rockville High School boys' varsity basketball team, then he is a talented athlete.

2. Todd plays on the Rockville High School boys' varsity basketball team.

Conclusion

Todd is a talented athlete.

Reasoning

To more easily recognize this pattern for deductive reasoning, we use letters to represent statements in the following generalization.

LAW OF DETACHMENT

Let P and Q represent simple statements, and assume that statements 1 and 2 are true. Then a valid argument having conclusion C has the form

$$\left.\begin{array}{l}\textbf{1.}\ \text{If } P\text{, then } Q\\ \textbf{2.}\ P\end{array}\right\}\ \text{premises}$$

$$\left.\text{C.}\ \therefore Q\right\}\ \text{conclusion}$$

NOTE: The symbol ∴ means "therefore."

In the preceding form, the statement "If *P*, then *Q*" is often read "*P* implies *Q*." That is, when *P* is known to be true, *Q* must follow.

Reasoning

We will use deductive reasoning throughout our work in geometry. For example, suppose that you know these two facts:

1. If an angle is a right angle, then it measures 90°.

2. Angle *A* is a right angle.

Because the form found in statements 1 and 2 matches the form of the valid argument, you may draw the following conclusion.

C. Angle *A* measures 90°.

VENN DIAGRAMS

Venn Diagrams

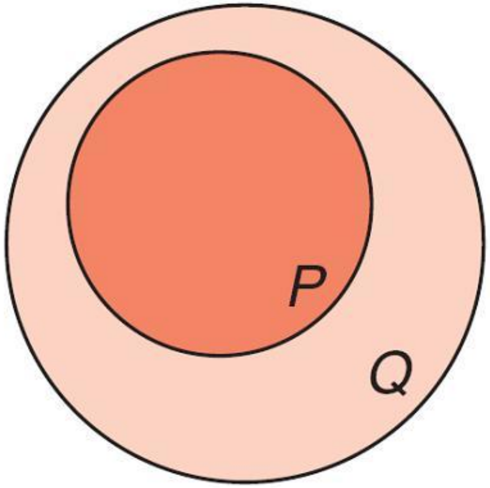
Sets of objects are often represented by geometric figures known as *Venn Diagrams*. Their creator, John Venn, was an Englishman who lived from 1834 to 1923.

In a Venn Diagram, each set is represented by a closed (bounded) figure such as a circle or rectangle.

If statements P and Q of the conditional statement “If P, then Q” are represented by sets of objects P and Q, respectively, then the Law of Detachment can be justified by a geometric argument.

Venn Diagrams

When a Venn Diagram is used to represent the statement "If *P*, then *Q*," it is absolutely necessary that circle *P* lies in circle *Q*; that is, *P* is a *subset* of *Q*. (See Figure 1.5.)



If *P*, then *Q*.

Figure 1.5

Example 11

Use Venn Diagrams to verify, if you accept the following statements 1 and 2 as true, what must you conclude?

1. If a student plays on the Rockville High School boys' varsity basketball team, then he is a talented athlete.

2. Todd plays on the Rockville High School boys' varsity basketball team.

Example 11 – *Solution*

Let *B* = students on the Rockville High varsity boys' basketball team.

Let *A* = people who are talented athletes.

To represent the statement "If a basketball player (*B*), then a talented athlete (*A*)," we show *B* within *A*.

In Figure 1.6 we use point *T* to represent Todd, a person on the basketball team (*T* in *B*). With point *T* also in circle *A*, we conclude that "Todd is a talented athlete."

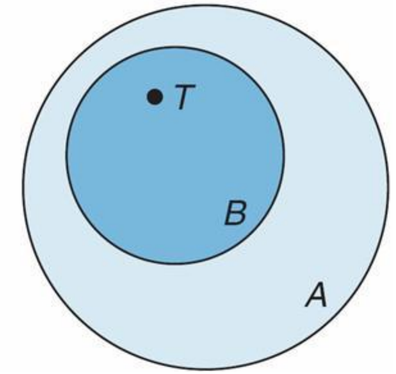


Figure 1.6

Venn Diagrams

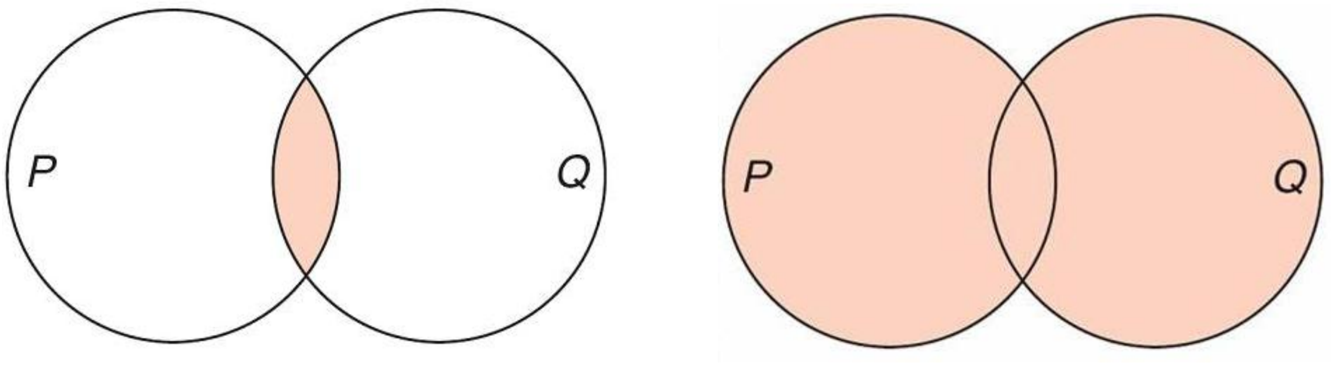
The statement "If *P*, then *Q*" is sometimes expressed in the form "All *P* are *Q*."

For instance, the conditional statement of Examples 8 and 11 can be written "All Rockville high school basketball players are talented athletes."

The compound statements known as the conjunction and the disjunction can also be related to the intersection and union of sets, relationships that can be illustrated by the use of Venn Diagrams.

Venn Diagrams

For the Venn Diagram, we assume that the sets *P* and *Q* may have elements in common. (See Figure 1.7.)



(a) $P \cap Q$ (b) $P \cup Q$

Figure 1.7

The elements common to *P* and *Q* form the **intersection** of *P* and *Q*, which is written $P \cap Q$.

Venn Diagrams

This set, $P \cap Q$, is the set of all elements in *both* P **and** Q. The elements that are in P, in Q, or in both form the **union** of P and Q, which is written $P \cup Q$.

This set, $P \cup Q$, is the set of elements in P **or** Q.