Quadratic Equations and Inequalities

CHAPTER 9

9.5 Properties of Quadratic Functions

Objectives

1 Graph a quadratic function

2 Find the x-intercepts of a parabola

Graph a quadratic function

Graph a quadratic function

QUADRATIC FUNCTION

A **quadratic function** is a function that can be expressed by the equation $f(x) = ax^2 + bx + c, a \neq 0$.

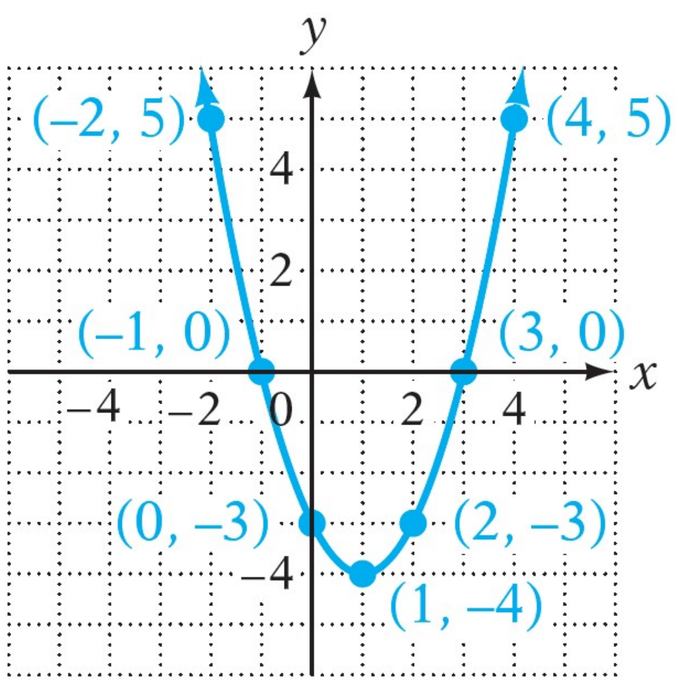
EXAMPLES

1. $f(x) = 2x^2 - 3x + 4$	$a = 2, b = -3, c = 4$
2. $g(x) = x^2 + 4x$	$a = 1, b = 4, c = 0$
3. $h(x) = 6 - x^2$	$a = -1, b = 0, c = 6$

The graph of a quadratic function can be drawn by finding ordered pairs that belong to the function.

Graph a quadratic function

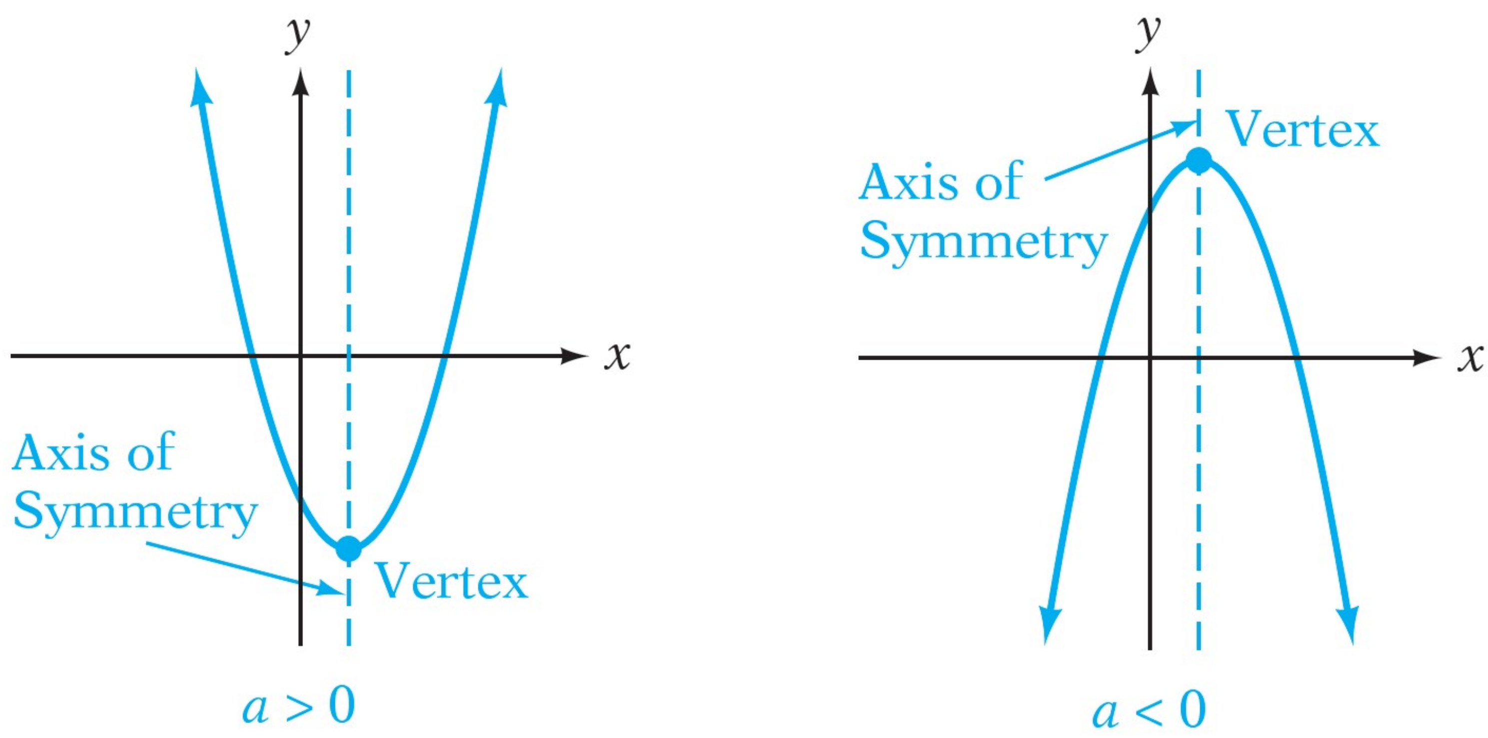
The graph of a quadratic function, called a **parabola**, is "cup" shaped, as shown below.



$f(x) = x^2 - 2x - 3$

Graph a quadratic function

The cup can open up or down. The parabola opens up when *a* > 0 and opens down when *a* < 0.



Graph a quadratic function

The **vertex** of a parabola is the point with the least *y*-coordinate when $a > 0$ and the point with the greatest *y*-coordinate when $a < 0$. The line that passes through the vertex and is parallel to the *y*-axis is called the **axis of symmetry**.

To understand the axis of symmetry, think of folding the graph along that line. The two halves of the graph would match up.

By following the process and completing the square of $f(x) = ax^2 + bx + c$, we can find a formula for the coordinates of the vertex of a parabola.

Graph a quadratic function

VERTEX AND AXIS OF SYMMETRY OF A PARABOLA

Let $f(x) = ax^2 + bx + c$, $a \neq 0$, be the equation of a parabola. The coordinates of the vertex are $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. The equation of the axis of symmetry is $x = -\frac{b}{2a}$.

EXAMPLE

Let $f(x) = -2x^2 + 8x + 5$. For this equation, $a = -2$, $b = 8$, and $c = 5$.

x-coordinate of the vertex: $-\frac{b}{2a} = -\frac{8}{2(-2)} = 2$

y-coordinate of the vertex: $f\left(-\frac{b}{2a}\right) = f(2) = -2(2)^2 + 8(2) + 5 = 13$

The coordinates of the vertex are $(2, 13)$.

The equation of axis of symmetry is $x = 2$.

Example 1

Find the coordinates of the vertex and the equation of the axis of symmetry for the parabola with equation $y = x^2 + 2x - 3$. Then sketch its graph.

Solution:

Find the *x*-coordinate of the vertex. *a* = 1 and *b* = 2.

$$x = -\frac{b}{2a} = -\frac{2}{2(1)} = -1$$

Find the *y*-coordinate of the vertex by replacing *x* with –1 and solving for *y*.

$$y = x^2 + 2x - 3$$

Example 1 – *Solution*

cont'd

$$y = (-1)^2 + 2(-1) - 3$$

$$y = 1 - 2 - 3 = -4$$

The coordinates of the vertex are (–1, –4). The equation of the axis of symmetry is $x = -1$.

Find some ordered-pair solutions of the equation and record these in a table. Because the graph is symmetric with respect to the line with equation $x = -1$, choose values of x greater than –1.

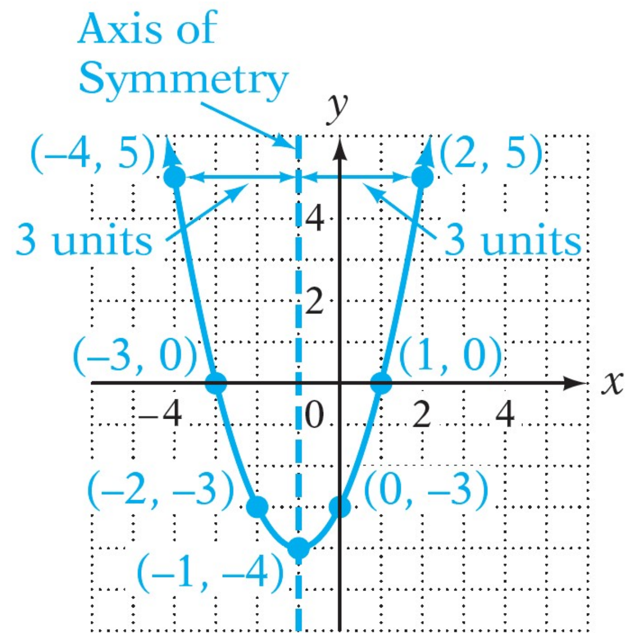
x	$y = x^2 + 2x - 3$
0	-3
1	0
2	5

Example 1 – *Solution*

cont'd

Graph the ordered-pair solutions on a rectangular coordinate system. Use symmetry to locate points of the graph on the other side of the axis of symmetry. Remember that corresponding points on the graph are the same distance from the axis of symmetry.

Draw a parabola through the points.



Graph a quadratic function

Because $f(x) = ax^2 + bx + c$ is a real number for all real numbers *x*, the domain of a quadratic function is all real numbers. The range of a quadratic function can be determined from the *y*-coordinate of the vertex.

Example 2

Graph $f(x) = -2x^2 - 4x + 3$. State the domain and range of *f*.

Solution:

Because *a* is negative ($a = -2$), the graph of *f* will open down. The *x*-coordinate of the vertex is

$$x = -\frac{b}{2a} = -\frac{-4}{2(-2)} = -1$$

The *y*-coordinate of the vertex is

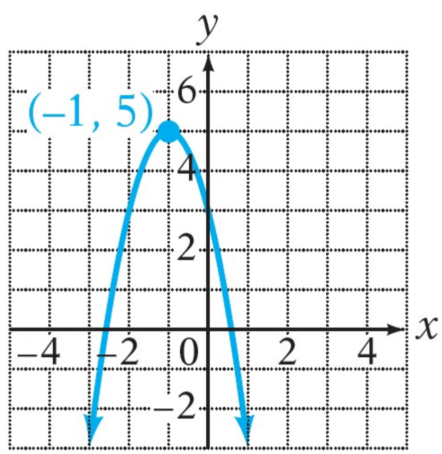
$$f(-1) = -2(-1)^2 - 4(-1) + 3 = 5$$

The coordinates of the vertex are $(-1, 5)$.

Example 2 – *Solution*

cont'd

Evaluate $f(x)$ for various values of x, and use symmetry to draw the graph.



Because $f(x) = -2x^2 - 4x + 3$ is a real number for all values of x, the domain of f is $\{x|x \in \text{real numbers}\}$. The vertex of the parabola is the highest point on the graph. Because the y-coordinate at that point is 5, the range of f is $\{y|y \leq 5\}$.

Find the *x*-intercepts of a parabola

Find the x-intercepts of a parabola

A point at which a graph crosses the x- or y-axis is called an *intercept* of the graph. The x-intercepts of the graph of an equation occur when $y = 0$; the y-intercepts occur when $x = 0$.

Example 3

Find the coordinates of the *x*-intercepts of the parabola whose equation is $y = 2x^2 - x - 6$.

Solution:

$$y = 2x^2 - x - 6$$

$$0 = 2x^2 - x - 6$$ **To find the *x*-intercepts, let $y = 0$.**

$$0 = (2x + 3)(x - 2)$$ **Factor.**

$$2x + 3 = 0 \qquad x - 2 = 0$$ **Use the Principle of Zero Products.**

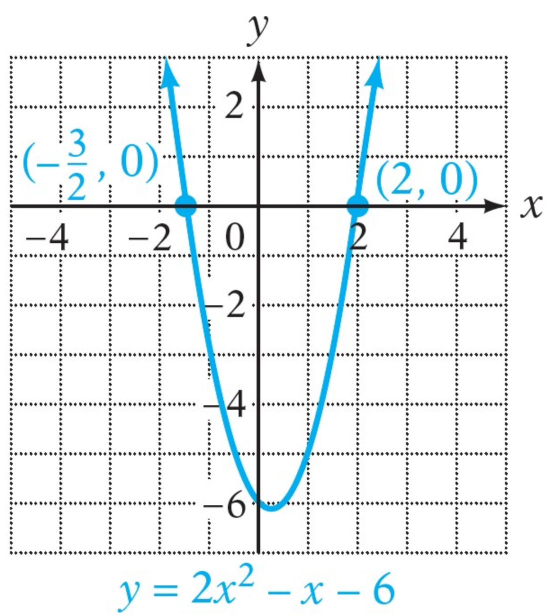
$$x = -\frac{3}{2} \qquad x = 2$$

Example 3 – *Solution*

cont'd

The coordinates of the x-intercepts are $(-\frac{3}{2}, 0)$ and $(2, 0)$.

See the graph below.



Find the *x*-intercepts of a parabola

If $ax^2 + bx + c = 0$ has a double root, then the graph of $y = ax^2 + bx + c$ intersects the *x*-axis at one point. In that case, the graph is said to be **tangent** to the *x*-axis.

A *zero* of a function *f* is a number *c* for which *f*(*c*) = 0.

There is an important connection between the *x*-intercepts and the real zeros of a function. Because the numbers on the *x*-axis are *real* numbers, the *x*-coordinate of an *x*-intercept of the graph of a function is a *real zero* of the function.

Example 4

Find the zeros of $f(x) = x^2 - 2x - 1$.

Solution:

$$f(x) = x^2 - 2x - 1$$

$$0 = x^2 - 2x - 1$$

To find the zeros, let $f(x) = 0$ and solve for x.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Use the quadratic formula.

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$

$a = 1$, $b = -2$, and $c = -1$.

$$= \frac{2 \pm \sqrt{4 + 4}}{2}$$

Example 4 – *Solution*

cont'd

$$= \frac{2 \pm \sqrt{8}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2}$$

$$= 1 \pm \sqrt{2}$$

The zeros of the function are $1 - \sqrt{2}$ and $1 + \sqrt{2}$.

The graph of $f(x) = x^2 - 2x - 1$ is shown at the left, along with the *x*-intercepts of the graph. Note that the *x*-coordinates of the *x*-intercepts are the real zeros of the function.

Find the *x*-intercepts of a parabola

The *x*-axis consists only of real numbers. If the graph of a function does not cross the *x*-axis, it will not have any *x*-intercepts and therefore will not have any *real number* zeros. In this case, the function has *complex number* zeros.

Example 5

Find the zeros of $f(x) = -x^2 + 2x - 2$.

Solution:

$$f(x) = -x^2 + 2x - 2$$

$$0 = -x^2 + 2x - 2$$

To find the zeros, let $f(x) = 0$ and solve for x.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Use the quadratic formula.

$$= \frac{-2 \pm \sqrt{2^2 - 4(-1)(-2)}}{2(-1)}$$

$a = -1, b = 2, c = -2$

Example 5 – *Solution*

cont'd

$$= \frac{-2 \pm \sqrt{4 - 8}}{-2}$$

$$= \frac{-2 \pm \sqrt{-4}}{-2}$$

$$= \frac{-2 \pm 2i}{-2}$$

$$= 1 \pm i$$

The complex zeros of the function are 1 – *i* and 1 + *i*.

The graph of $f(x) = -x^2 + 2x - 2$ is shown at the left. Note that the graph does not cross the *x*-axis. The function has no real zeros.

Find the *x*-intercepts of a parabola

Example 5 shows that a function can have a zero without crossing the *x*-axis. The zero of the function, in this case, is a complex number.

If the graph of a function crosses the *x*-axis, the function has a real zero at the *x*-coordinate of the *x*-intercept. If the graph of a function never crosses the *x*-axis, the function has only complex number zeros.

Find the *x*-intercepts of a parabola

The discriminant of $ax^2 + bx + c$ is the expression $b^2 - 4ac$ and that this expression can be used to determine whether $ax^2 + bx + c = 0$ has zero, one, or two real number solutions. Because there is a connection between the solutions of $ax^2 + bx + c = 0$ and the *x*-intercepts of the graph of $y = ax^2 + bx + c$, the discriminant can be used to determine the number of *x*-intercepts of a parabola.

EFFECT OF THE DISCRIMINANT ON THE NUMBER OF *x*-INTERCEPTS OF A PARABOLA WITH EQUATION $y = ax^2 + bx + c$

1. If $b^2 - 4ac = 0$, the parabola has one x-intercept.
2. If $b^2 - 4ac > 0$, the parabola has two x-intercepts.
3. If $b^2 - 4ac < 0$, the parabola has no x-intercepts.

Example 6

Use the discriminant to determine the number of *x*-intercepts of the parabola with the given equation.

A. $y = 2x^2 - x + 2$ **B.** $f(x) = -x^2 + 4x - 4$

Solution:

A. $y = 2x^2 - x + 2$ Evaluate the discriminant.

$b^2 - 4ac$ $a = 2, b = -1, c = 2$

$(-1)^2 - 4(2)(2) = 1 - 16$

$= -15$

The discriminant is negative.

The parabola has no *x*-intercepts.

Example 6 – *Solution*

cont'd

B. $f(x) = -x^2 + 4x - 4$

$$b^2 - 4ac$$ Evaluate the discriminant.

$$(4)^2 - 4(-1)(-4) = 16 - 16$$ $a = -1, b = 4, c = -4$

$$= 0$$

The discriminant is zero.

The parabola has one *x*-intercept.