CHAPTER 10

Functions and Relations

10.3 One-to-One and Inverse Functions

Objectives

1 Determine whether a function is one-to-one

2 Find the inverse of a function

Determine whether a function is one-to-one

Determine whether a function is one-to-one

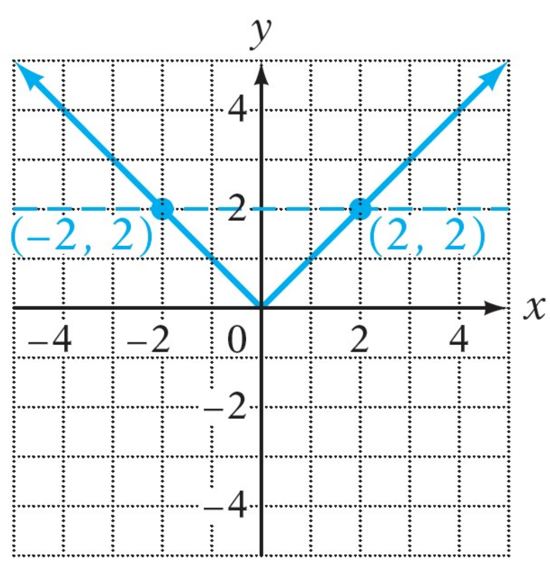
A function is a set of ordered pairs in which no two ordered pairs that have the same first component have different second components.

This means that given any x, there is only one y that can be paired with that x.

A **one-to-one function** satisfies the additional condition that given any y, there is only one x that can be paired with the given y. One-to-one functions are commonly expressed by writing 1–1.

Determine whether a function is one-to-one

The function given by the equation $y = |x|$ is not a 1–1 function since, given $y = 2$, there are two possible values of x, 2 and –2, that can be paired with the given y-value. The graph shown below illustrates that a horizontal line intersects the graph more than once.



Determine whether a function is one-to-one

Just as the vertical line test can be used to determine whether a graph represents a function, a **horizontal line test** can be used to determine whether the graph of a function represents a 1–1 function.

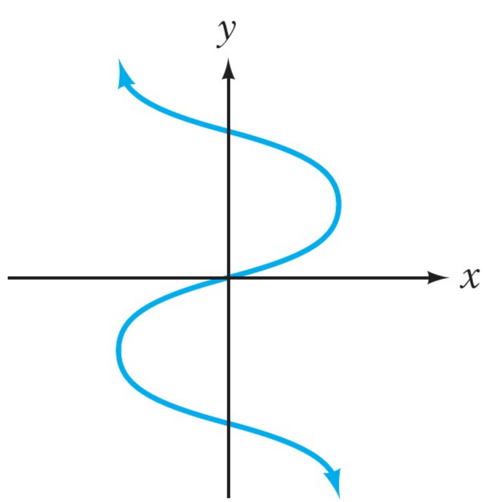
HORIZONTAL LINE TEST

A graph of a function is the graph of a 1–1 function if any horizontal line intersects the graph at no more than one point.

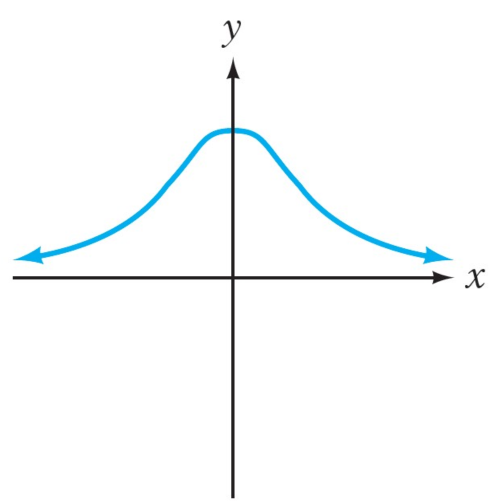
Example 1

Determine whether the graph represents the graph of a 1–1 function.

A.

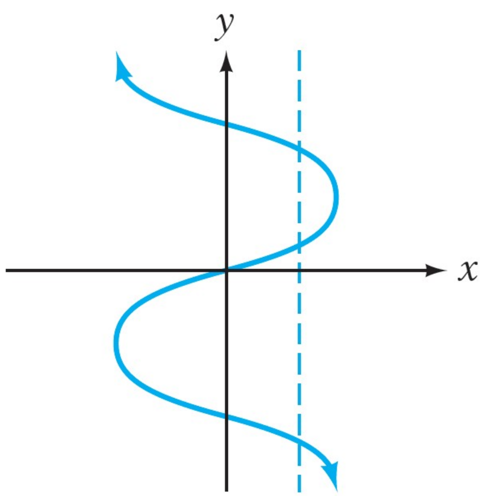


B.



Example 1 – *Solution*

A.



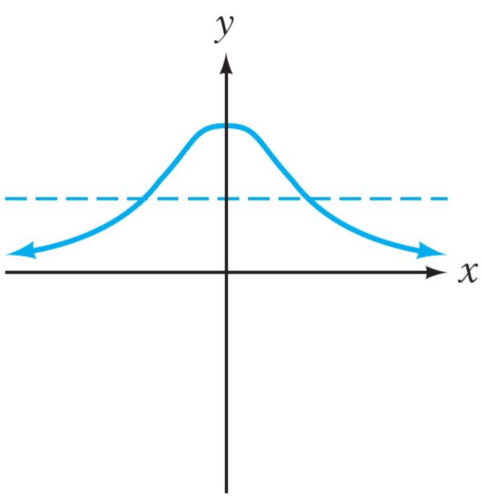
A vertical line can intersect the graph at more than one point. The graph does not represent a function.

This is not the graph of a 1–1 function.

Example 1 – *Solution*

cont'd

B.



A horizontal line can intersect the curve at more than one point.

This is not the graph of a 1–1 function.

Find the inverse of a function

Find the inverse of a function

The **inverse of a function** is the set of ordered pairs formed by reversing the coordinates of each ordered pair of the function.

Note that the domain of the inverse function is the range of the original function, and the range of the inverse function is the domain of the original function.

Now consider the function defined by $g(x) = x^2$ with domain $\{-2, -1, 0, 1, 2\}$. The set of ordered pairs of this function is $\{(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)\}$.

Find the inverse of a function

Reversing the coordinates of the ordered pairs gives {(4, –2), (1, –1), (0, 0), (1, 1), (4, 2)}.

These ordered pairs do not satisfy the condition of a function because there are ordered pairs with the same first coordinate and different second coordinates.

This example illustrates that not all functions have an inverse function.

Find the inverse of a function

CONDITION FOR AN INVERSE FUNCTION

A function f has an inverse function if and only if f is a 1–1 function.

The symbol f^{-1} is used to denote the inverse of a 1–1 function *f*. The symbol $f^{-1}(x)$ is read "*f* inverse of *x*." $f^{-1}(x)$ does not denote the reciprocal of $f(x)$ but is the notation for the inverse of a 1–1 function.

To find the inverse of a function, interchange *x* and *y*. Then solve for *y*.

Example 2

Find the inverse of the function defined by the equation $f(x) = 2x - 4$.

Solution:

$$f(x) = 2x - 4$$

$$y = 2x - 4 \qquad \textbf{Replace } f(x) \textbf{ by } y.$$

$$x = 2y - 4 \qquad \textbf{Interchange } x \textbf{ and } y.$$

$$2y = x + 4 \qquad \textbf{Solve for } y.$$

$$y = \frac{1}{2}x + 2$$

$$f^{-1}(x) = \frac{1}{2}x + 2$$

Find the inverse of a function

If two functions are inverses of each other, then their graphs are mirror images with respect to the graph of the equation $y = x$.

A special property relates the composition of a function and its inverse.

COMPOSITION OF INVERSE FUNCTIONS PROPERTY

$$f^{-1}[f(x)] = x \qquad \text{and} \qquad f[f^{-1}(x)] = x$$

This property can be used to determine whether two functions are inverses of each other.

Example 3

Are the functions defined by the equations $f(x) = -2x + 3$ and $g(x) = -\frac{1}{2}x + \frac{3}{2}$ inverses of each other?

Solution:

$$f[g(x)] = f\left(-\frac{1}{2}x + \frac{3}{2}\right)$$

Check that the functions *f* and *g* satisfy the Composition of Inverse Functions Property.

$$= -2\left(-\frac{1}{2}x + \frac{3}{2}\right) + 3$$

$$= x - 3 + 3$$

$$= x$$

$f[g(x)] = x$

Example 3 – *Solution*

cont'd

$$g[f(x)] = g(-2x + 3)$$

$$= -\frac{1}{2}(-2x + 3) + \frac{3}{2}$$

$$= x - \frac{3}{2} + \frac{3}{2}$$

$$= x$$

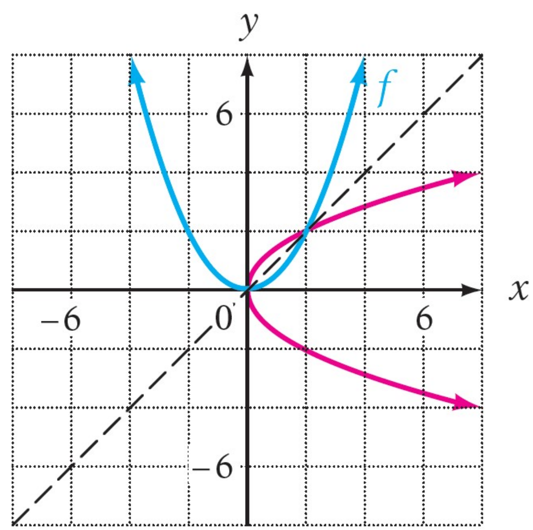
$\boldsymbol{g[f(x)] = x}$

The functions are inverses of each other.

Find the inverse of a function

The function given by the equation $f(x) = \frac{1}{2}x^2$ does not have an inverse that is a function. Two of the ordered-pair solutions of this function are (4, 8) and (–4, 8).

The graph of $f(x) = \frac{1}{2}x^2$ is shown below.



Find the inverse of a function

This graph does not pass the horizontal line test for the graph of a 1–1 function.

The mirror image of the graph of *f* with respect to the graph of $y = x$ is also shown. This graph does not pass the vertical line test for the graph of a function.

A quadratic function with the real numbers as its domain does not have an inverse function.