CHAPTER 4

Systems of Equations and Inequalities

4.3 Solving Systems of Equations by Using Determinants and by Using Matrices

Objectives

1 Evaluate determinants

2 Solve systems of linear equations by using Cramer's Rule

3 Solve systems of linear equations by using matrices

Evaluate determinants

Evaluate determinants

A **matrix** is a rectangular array of numbers. Each number in the matrix is called an **element** of the matrix. The matrix at the right, with three rows and four columns, is called a 3×4 ("3 by 4") matrix.

$$A = \begin{bmatrix} 1 & -3 & 2 & 4 \\ 0 & 4 & -3 & 2 \\ 6 & -5 & 4 & -1 \end{bmatrix}$$

A matrix of m rows and n columns is said to be of **order** $\boldsymbol{m} \times \boldsymbol{n}$. Matrix A above has order 3×4. The notation $\boldsymbol{a_{ij}}$ refers to the element of a matrix in the ith row and the jth column.

For matrix A, $a_{23} = -3$, $a_{31} = 6$, and $a_{13} = 2$.

Evaluate determinants

A **square matrix** is one that has the same number of rows as columns. A 2×2 matrix and a 3×3 matrix are shown at the right.

$$\begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix} \quad \begin{bmatrix} 4 & 0 & 1 \\ 5 & -3 & 7 \\ 2 & 1 & 4 \end{bmatrix}$$

Associated with every square matrix is a number called its **determinant.**

Evaluate determinants

DETERMINANT OF A 2 × 2 MATRIX

The determinant of a 2×2 matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is written $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$. The value of this determinant is given by the formula

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

EXAMPLES

1. Evaluate the determinant $\begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix}$.

$$\begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} = 3(2) - 4(-1) = 6 + 4 = 10$$

2. Evaluate the determinant $\begin{vmatrix} -3 & -5 \\ 6 & 10 \end{vmatrix}$.

$$\begin{vmatrix} -3 & -5 \\ 6 & 10 \end{vmatrix} = -3(10) - (-5)(6) = -30 + 30 = 0$$

Evaluate determinants

The value of the determinant of a matrix of order larger than 2×2 can be found by using smaller determinants within the large one. The smaller determinants are called *minors*.

Evaluate determinants

MINOR OF AN ELEMENT OF A DETERMINANT

The **minor of an element** a_{ij} of a determinant is the determinant that remains after row i and column j have been removed.

EXAMPLES

1. Find the minor of -3 for the determinant $\begin{vmatrix} 2 & -3 & 4 \\ 0 & 4 & 8 \\ -1 & 3 & 6 \end{vmatrix}$.

$$\begin{vmatrix} 2 & -3 & 4 \\ 0 & 4 & 8 \\ -1 & 3 & 6 \end{vmatrix}$$

- **Remove the row and column containing -3.**

The minor of -3 is $\begin{vmatrix} 0 & 8 \\ -1 & 6 \end{vmatrix}$.

2. Find the minor of 0 for the determinant $\begin{vmatrix} 2 & -3 & 3 & 1 \\ 5 & -2 & 1 & -7 \\ 4 & 7 & 0 & 9 \\ 3 & -1 & 6 & 10 \end{vmatrix}$.

$$\begin{vmatrix} 2 & -3 & 3 & 1 \\ 5 & -2 & 1 & -7 \\ 4 & 7 & 0 & 9 \\ 3 & -1 & 6 & 10 \end{vmatrix}$$

- **Remove the row and column containing 0.**

The minor of 0 is $\begin{vmatrix} 2 & -3 & 1 \\ 5 & -2 & -7 \\ 3 & -1 & 10 \end{vmatrix}$.

Evaluate determinants

Related to the minor of an element of a determinant is the *cofactor* of the element.

DEFINITION OF A COFACTOR

The **cofactor of an element** a_{ij} of a determinant is $(-1)^{i+j}$ times the minor of a_{ij}.

EXAMPLES

Use the determinant $\begin{vmatrix} 3 & -2 & 1 \\ 2 & -5 & -4 \\ 0 & 3 & 1 \end{vmatrix}$.

1. Find the cofactor of -2.
Because -2 is in the first row and the second column, $i = 1$ and $j = 2$. Therefore, $i + j = 1 + 2 = 3$, and $(-1)^{i+j} = (-1)^3 = -1$.

The cofactor of -2 is $(-1)\begin{vmatrix} 2 & -4 \\ 0 & 1 \end{vmatrix}$.

2. Find the cofactor of -5.
Because -5 is in the second row and the second column, $i = 2$ and $j = 2$. Therefore, $i + j = 2 + 2 = 4$, and $(-1)^{i+j} = (-1)^4 = 1$.

The cofactor of -5 is $(1)\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix}$.

Evaluate determinants

Note from these examples that the cofactor of an element is –1 times the minor of that element or 1 times the minor of that element, depending on whether the sum $i + j$ is an odd or an even integer.

The value of a 3×3 or larger determinant can be found by **expanding by cofactors** of *any* row or *any* column.

This process involves multiplying each element of the selected row or column by its cofactor and then finding the sum of the results.

Evaluate determinants

Note that the value of the determinant is the same whether the first row or the second column is used to expand by cofactors.

Any row or column can be used to evaluate a determinant by expanding by cofactors. In order to simplify the calculations, choose the row or column with the most zeros.

Example 1

Evaluate the determinant.

A. $\begin{vmatrix} 3 & -2 \\ 6 & -4 \end{vmatrix}$ **B.** $\begin{vmatrix} -2 & 3 & 1 \\ 4 & -2 & 0 \\ 1 & -2 & 3 \end{vmatrix}$

Solution:

A. $\begin{vmatrix} 3 & -2 \\ 6 & -4 \end{vmatrix} = 3(-4) - (6)(-2)$

$= -12 + 12$

$= 0$

Example 1 – *Solution*

cont'd

B. There is a zero in row 2, column 3. Expand by cofactors of either row 2 or column 3. We will use row 2.

$$\begin{vmatrix} -2 & 3 & 1 \\ 4 & -2 & 0 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= 4(-1)^{2+1}\begin{vmatrix} 3 & 1 \\ -2 & 3 \end{vmatrix} + (-2)(-1)^{2+2}\begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix} + 0(-1)^{2+3}\begin{vmatrix} -2 & 3 \\ 1 & -2 \end{vmatrix}$$

$$= 4(-1)\begin{vmatrix} 3 & 1 \\ -2 & 3 \end{vmatrix} + (-2)(1)\begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix} + 0(-1)\begin{vmatrix} -2 & 3 \\ 1 & -2 \end{vmatrix}$$

$$= -4(9 - (-2)) - 2(-6 - 1) + 0$$

$$= -4(11) - 2(-7) = -44 + 14$$

$$= -30$$

Solve systems of linear equations by using Cramer’s Rule

Solve systems of linear equations by using Cramer's Rule

The connection between determinants and systems of equations can be understood by solving a general system of linear equations.

Solve:

$$a_{11}x + a_{12}y = b_1 \quad (1)$$
$$a_{21}x + a_{22}y = b_2 \quad (2)$$

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_2$$

Eliminate *y*. Multiply equation (1) by a_{22} and equation (2) by $-a_{12}$.

Add the equations.

$$\begin{array}{r} a_{11}a_{22}x + a_{12}a_{22}y = b_1a_{22} \\ -a_{21}a_{12}x - a_{12}a_{22}y = -b_2a_{12} \\ \hline (a_{11}a_{22} - a_{21}a_{12})x = b_1a_{22} - b_2a_{12} \end{array}$$

Solve systems of linear equations by using Cramer's Rule

Solve for *x*. Assume $a_{11}a_{22} - a_{21}a_{12} \neq 0.$

$$x = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}}$$

Note that the denominator for *x*, $a_{11}a_{22} - a_{21}a_{12}$, is the determinant of the coefficients of *x* and *y*. This is called the **coefficient determinant.**

$$a_{11}a_{22} - a_{21}a_{12} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Coefficients of x

Coefficients of y

Solve systems of linear equations by using Cramer's Rule

The numerator for x, $b_1a_{22} - b_2a_{12}$, is the determinant obtained by replacing the first column in the coefficient determinant by the constants b_1 and b_2.

$$b_1a_{22} - b_2a_{12} = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}$$

Constants of the equations

This is called the **numerator determinant**.

Following a similar procedure and eliminating x, we can also express the y-coordinate of the solution in determinant form. These results are summarized in Cramer's Rule.

Solve systems of linear equations by using Cramer's Rule

CRAMER'S RULE FOR TWO EQUATIONS IN TWO VARIABLES

The solution of the system of equations

$$\begin{aligned} a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2 \end{aligned}$$

is given by $x = \dfrac{D_x}{D}$ and $y = \dfrac{D_y}{D}$, where

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, D_x = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}, D_y = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}, \text{ and } D \neq 0.$$

Example 2

Solve by using Cramer's Rule:

$$2x - 3y = 8$$
$$5x + 6y = 11$$

Solution:

$$D = \begin{vmatrix} 2 & -3 \\ 5 & 6 \end{vmatrix} = 27$$

Evaluate the coefficient determinant.

$$D_x = \begin{vmatrix} 8 & -3 \\ 11 & 6 \end{vmatrix} = 81$$

Evaluate each numerator determinant.

$$D_y = \begin{vmatrix} 2 & 8 \\ 5 & 11 \end{vmatrix} = -18$$

$$x = \frac{D_x}{D} = \frac{81}{27} = 3 \qquad y = \frac{D_y}{D} = \frac{-18}{27} = -\frac{2}{3}$$

Use Cramer's Rule to find the *x*- and *y*-coordinates of the solution.

The solution is $\left(3, -\frac{2}{3}\right)$

Solve systems of linear equations by using Cramer's Rule

A procedure similar to that followed for two equations in two variables can be used to extend Cramer's Rule to three equations in three variables.

CRAMER'S RULE FOR THREE EQUATIONS IN THREE VARIABLES

The solution of the system of equations

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

is given by $x = \dfrac{D_x}{D}$, $y = \dfrac{D_y}{D}$, and $z = \dfrac{D_z}{D}$, where

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, D_x = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, D_y = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix},$$

$$D_z = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}, \text{ and } D \neq 0.$$

Example 3

Solve by using Cramer's Rule:

$$3x - y + z = 5$$
$$x + 2y - 2z = -3$$
$$2x + 3y + z = 4$$

Solution:

$$D = \begin{vmatrix} 3 & -1 & 1 \\ 1 & 2 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 28$$

Evaluate the coefficient determinant.

$$D_x = \begin{vmatrix} 5 & -1 & 1 \\ -3 & 2 & -2 \\ 4 & 3 & 1 \end{vmatrix} = 28$$

Evaluate each numerator determinant.

$$D_y = \begin{vmatrix} 3 & 5 & 1 \\ 1 & -3 & -2 \\ 2 & 4 & 1 \end{vmatrix} = 0$$

Example 3 – *Solution*

cont'd

$$D_z = \begin{vmatrix} 3 & -1 & 5 \\ 1 & 2 & -3 \\ 2 & 3 & 4 \end{vmatrix} = 56$$

$$x = \frac{D_x}{D} = \frac{28}{28} = 1$$

$$y = \frac{D_y}{D} = \frac{0}{28} = 0$$

$$z = \frac{D_z}{D} = \frac{56}{28} = 2$$

Use Cramer's Rule to find the *x*-, *y*-, and *z*-coordinates of the solution.

The solution is (1, 0, 2)

Solve systems of linear equations by using matrices

Solve systems of linear equations by using matrices

Consider the 3×4 matrix below.

$$\begin{bmatrix} 1 & 4 & -3 & 6 \\ -2 & 5 & 2 & 0 \\ -1 & 3 & 7 & -4 \end{bmatrix}$$

The elements a_{11}, a_{22}, a_{33}, ... , a_{nn} form the **main diagonal** of a matrix. The elements 1, 5, and 7 form the main diagonal of the matrix above.

Solve systems of linear equations by using matrices

By considering only the coefficients and constants for the following system of equations, we can form the corresponding 3 × 4 **augmented matrix**.

System of Equations

$$\begin{aligned} 3x - 2y + z &= 2 \\ x - 3z &= -2 \\ 2x - y + 4z &= 5 \end{aligned}$$

Augmented Matrix

$$\left[\begin{array}{rrr|r} 3 & -2 & 1 & 2 \\ 1 & 0 & -3 & -2 \\ 2 & -1 & 4 & 5 \end{array}\right]$$

Example 4

Write the augmented matrix for the system of equations.

$$\begin{aligned} 2x - 3y &= 4 \\ x + 5y &= 0 \end{aligned}$$

Solution:

The coefficients of x, 2 and 1, are the first column. The coefficients of y, –3 and 5, are the second column. The constant terms are the third column.

The augmented matrix is $\left[\begin{array}{rr|r} 2 & -3 & 4 \\ 1 & 5 & 0 \end{array}\right]$.

Solve systems of linear equations by using matrices

A system of equations can be solved by writing the system in matrix form and then performing operations on the matrix similar to those performed on the equations of the system.

These operations are called **elementary row operations.**

ELEMENTARY ROW OPERATIONS

1. Interchange two rows.
2. Multiply all the elements in a row by the same nonzero number.
3. Replace a row by the sum of that row and a multiple of any other row.

Example 5

Let

$$A = \left[\begin{array}{rrr|r} 1 & 3 & -4 & 6 \\ 3 & 2 & 0 & -1 \\ -2 & -5 & 3 & 4 \end{array}\right].$$

Perform the following elementary row operations on *A*.

A. $R_1 \leftrightarrow R_3$ **B.** $-2R_3$ **C.** $2R_3 + R_1$

Solution:

A. $R_1 \leftrightarrow R_3$ means to interchange row 1 and row 3.

$$\left[\begin{array}{rrr|r} 1 & 3 & -4 & 6 \\ 3 & 2 & 0 & -1 \\ -2 & -5 & 3 & 4 \end{array}\right] \quad R_1 \leftrightarrow R_3 \quad \left[\begin{array}{rrr|r} -2 & -5 & 3 & 4 \\ 3 & 2 & 0 & -1 \\ 1 & 3 & -4 & 6 \end{array}\right]$$

Example 5 – *Solution*

cont'd

B. $-2R_3$ means to multiply row 3 by –2.

$$\left[\begin{array}{rrr|r} 1 & 3 & -4 & 6 \\ 3 & 2 & 0 & -1 \\ -2 & -5 & 3 & 4 \end{array}\right] \quad -2R_3 \rightarrow \quad \left[\begin{array}{rrr|r} 1 & 3 & -4 & 6 \\ 3 & 2 & 0 & -1 \\ 4 & 10 & -6 & -8 \end{array}\right]$$

C. $2R_3 + R_1$ means to multiply row 3 by 2 and then add the result to row 1. The result replaces row 1.

$$\left[\begin{array}{rrr|r} 1 & 3 & -4 & 6 \\ 3 & 2 & 0 & -1 \\ -2 & -5 & 3 & 4 \end{array}\right] \quad 2R_3 + R_1 \rightarrow \quad \left[\begin{array}{rrr|r} -3 & -7 & 2 & 14 \\ 3 & 2 & 0 & -1 \\ -2 & -5 & 3 & 4 \end{array}\right]$$

Solve systems of linear equations by using matrices

Elementary row operations are used to solve a system of equations. The goal is to use the elementary row operations to rewrite the augmented matrix with 1's down the main diagonal and 0's to the left of the 1's in all rows except the first.

This is called the **row echelon form** of the matrix. Here are some examples of the row echelon form.

$$\left[\begin{array}{cc|c} 1 & 3 & -2 \\ 0 & 1 & 3 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & 2.5 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 4 & \frac{1}{2} & -3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} \end{array}\right]$$

Solve systems of linear equations by using matrices

We will follow a very definite procedure to rewrite an augmented matrix in row echelon form.

STEPS FOR REWRITING A 2 × 3 AUGMENTED MATRIX IN ROW ECHELON FORM

The order in which the elements of the augmented matrix below are changed is as follows:

Step 1: Change a_{11} to a 1.

Step 2: Change a_{21} to a 0.

Step 3: Change a_{22} to a 1.

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array}\right]$$

Solve systems of linear equations by using matrices

The row echelon form of a matrix is not unique and depends on the elementary row operations that are used. For instance, suppose we again start with $\left[\begin{array}{rr|r} 3 & -6 & 12 \\ 2 & 1 & -3 \end{array}\right]$ and follow the elementary row operations below.

$$\left[\begin{array}{rr|r} 3 & -6 & 12 \\ 2 & 1 & -3 \end{array}\right] \; -1R_2 + R_1 \rightarrow \left[\begin{array}{rr|r} 1 & -7 & 15 \\ 2 & 1 & -3 \end{array}\right] \; -2R_1 + R_2 \rightarrow \left[\begin{array}{rr|r} 1 & -7 & 15 \\ 0 & 15 & -33 \end{array}\right] \; \tfrac{1}{15}R_2 \rightarrow \left[\begin{array}{rr|r} 1 & -7 & 15 \\ 0 & 1 & -\frac{11}{5} \end{array}\right]$$

The row echelon forms $\left[\begin{array}{rr|r} 1 & -7 & 15 \\ 0 & 1 & -\frac{11}{5} \end{array}\right]$ and $\left[\begin{array}{rr|r} 1 & -2 & 4 \\ 0 & 1 & -\frac{11}{5} \end{array}\right]$ are different.

Row echelon form is not unique.

Solve systems of linear equations by using matrices

STEPS FOR REWRITING A 3 × 4 AUGMENTED MATRIX IN ROW ECHELON FORM

The order in which the elements of the augmented matrix below are changed is as follows:

Step 1: Change a_{11} to a 1.

Step 2: Change a_{21} and a_{31} to 0's.

Step 3: Change a_{22} to a 1.

Step 4: Change a_{32} to a 0.

Step 5: Change a_{33} to a 1.

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]$$

Example 6

Write $\left[\begin{array}{rrr|r} 2 & 1 & 3 & -1 \\ 1 & 3 & 5 & -1 \\ -3 & -1 & 1 & 2 \end{array}\right]$ in row echelon form.

Solution:

Step 1: Change a_{11} to 1 by interchanging row 1 and row 2.

Note: We could have chosen to multiply row 1 by 1/2. The sequence of steps to get to row echelon form is not unique.

$$\left[\begin{array}{rrr|r} 2 & 1 & 3 & -1 \\ 1 & 3 & 5 & -1 \\ -3 & -1 & 1 & 2 \end{array}\right] \quad R_1 \leftrightarrow R_2 \quad \left[\begin{array}{rrr|r} 1 & 3 & 5 & -1 \\ 2 & 1 & 3 & -1 \\ -3 & -1 & 1 & 2 \end{array}\right]$$

Example 6 – *Solution*

cont'd

Step 2: Change a_{21} to 0 by multiplying row 1 by the opposite of a_{21} and then adding to row 2.

$$\left[\begin{array}{rrr|r} 1 & 3 & 5 & -1 \\ 2 & 1 & 3 & -1 \\ -3 & -1 & 1 & 2 \end{array}\right] \quad -2R_1 + R_2 \rightarrow \quad \left[\begin{array}{rrr|r} 1 & 3 & 5 & -1 \\ 0 & -5 & -7 & 1 \\ -3 & -1 & 1 & 2 \end{array}\right]$$

Change a_{31} to 0 by multiplying row 1 by the opposite of a_{31} and then adding to row 3.

$$\left[\begin{array}{rrr|r} 1 & 3 & 5 & -1 \\ 0 & -5 & -7 & 1 \\ -3 & -1 & 1 & 2 \end{array}\right] \quad 3R_1 + R_3 \rightarrow \quad \left[\begin{array}{rrr|r} 1 & 3 & 5 & -1 \\ 0 & -5 & -7 & 1 \\ 0 & 8 & 16 & -1 \end{array}\right]$$

Example 6 – *Solution*

cont'd

Step 3: Change a_{22} to 1 by multiplying row 2 by the reciprocal of a_{22}.

$$\left[\begin{array}{rrr|r} 1 & 3 & 5 & -1 \\ 0 & -5 & -7 & 1 \\ 0 & 8 & 16 & -1 \end{array}\right] \quad -\tfrac{1}{5}R_2 \rightarrow \quad \left[\begin{array}{rrr|r} 1 & 3 & 5 & -1 \\ 0 & 1 & \frac{7}{5} & -\frac{1}{5} \\ 0 & 8 & 16 & -1 \end{array}\right]$$

Step 4: Change a_{32} to 0 by multiplying row 2 by the opposite of a_{32} and then adding to row 3.

$$\left[\begin{array}{rrr|r} 1 & 3 & 5 & -1 \\ 0 & 1 & \frac{7}{5} & -\frac{1}{5} \\ 0 & 8 & 16 & -1 \end{array}\right] \quad -8R_2 + R_3 \rightarrow \quad \left[\begin{array}{rrr|r} 1 & 3 & 5 & -1 \\ 0 & 1 & \frac{7}{5} & -\frac{1}{5} \\ 0 & 0 & \frac{24}{5} & \frac{3}{5} \end{array}\right]$$

Example 6 – *Solution*

cont'd

Step 5: Change a_{33} to 1 by multiplying row 3 by the reciprocal of a_{33}.

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & -1 \\ 0 & 1 & \frac{7}{5} & -\frac{1}{5} \\ 0 & 0 & \frac{24}{5} & \frac{3}{5} \end{array}\right] \quad \frac{5}{24}R_3 \rightarrow \quad \left[\begin{array}{ccc|c} 1 & 3 & 5 & -1 \\ 0 & 1 & \frac{7}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{8} \end{array}\right]$$

A row echelon form of the matrix is $\left[\begin{array}{ccc|c} 1 & 3 & 5 & -1 \\ 0 & 1 & \frac{7}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{8} \end{array}\right]$.

Solve systems of linear equations by using matrices

If an augmented matrix is in row echelon form, the corresponding system of equations can be solved by substitution.

The process of solving a system of equations by using elementary row operations is called the **Gaussian elimination method.**

The Gaussian elimination method can be extended to systems of equations with more than two variables.

Example 8

Solve by using the Gaussian elimination method:

$$\begin{aligned} x + 2y - z &= 9 \\ 2x - y + 2z &= -1 \\ -2x + 3y - 2z &= 7 \end{aligned}$$

Solution:

Write the augmented matrix and then use elementary row operations to rewrite the matrix in row echelon form.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 9 \\ 2 & -1 & 2 & -1 \\ -2 & 3 & -2 & 7 \end{array}\right] \xrightarrow[-2R_1 + R_2]{\substack{a_{11} \text{ is 1. Change} \\ a_{21} \text{ to 0.}}} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 9 \\ 0 & -5 & 4 & -19 \\ -2 & 3 & -2 & 7 \end{array}\right]$$

Example 8 – *Solution*

cont'd

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 9 \\ 0 & -5 & 4 & -19 \\ -2 & 3 & -2 & 7 \end{array}\right] \xrightarrow[2R_1 + R_3]{\text{Change } a_{31} \text{ to } 0.} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 9 \\ 0 & -5 & 4 & -19 \\ 0 & 7 & -4 & 25 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 9 \\ 0 & -5 & 4 & -19 \\ 0 & 7 & -4 & 25 \end{array}\right] \xrightarrow[-\frac{1}{5}R_2]{\text{Change } a_{22} \text{ to } 1.} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 9 \\ 0 & 1 & -\frac{4}{5} & \frac{19}{5} \\ 0 & 7 & -4 & 25 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 9 \\ 0 & 1 & -\frac{4}{5} & \frac{19}{5} \\ 0 & 7 & -4 & 25 \end{array}\right] \xrightarrow[-7R_2 + R_3]{\text{Change } a_{32} \text{ to } 0.} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 9 \\ 0 & 1 & -\frac{4}{5} & \frac{19}{5} \\ 0 & 0 & \frac{8}{5} & -\frac{8}{5} \end{array}\right]$$

Example 8 – *Solution*

cont'd

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 0 & 1 & -\frac{4}{5} & \frac{19}{5} \\ 0 & 0 & \frac{8}{5} & -\frac{8}{5} \end{array}\right] \xrightarrow[\frac{5}{8}R_2]{\text{Change } a_{33} \text{ to } 1.} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 0 & 1 & -\frac{4}{5} & \frac{19}{5} \\ 0 & 0 & 1 & -1 \end{array}\right]$$

This is row echelon form.

$$(1) \qquad x + 2y - z = 9$$

$$(2) \qquad y - \frac{4}{5}z = \frac{19}{5}$$

$$(3) \qquad z = -1$$

Write the system of equations corresponding to the row echelon form of the matrix.

$$y - \frac{4}{5}(-1) = \frac{19}{5}$$

$$y + \frac{4}{5} = \frac{19}{5}$$

Substitute -1 for z in equation (2) and solve for y.

Example 8 – *Solution*

cont'd

$$y = 3$$

$$x + 2y - z = 9$$

$$x + 2(3) - (-1) = 9$$

$$x + 7 = 9$$

$$x = 2$$

Substitute −1 for *z* and 3 for *y* in equation (1) and solve for *x*.

The solution is (2, 3, –1).