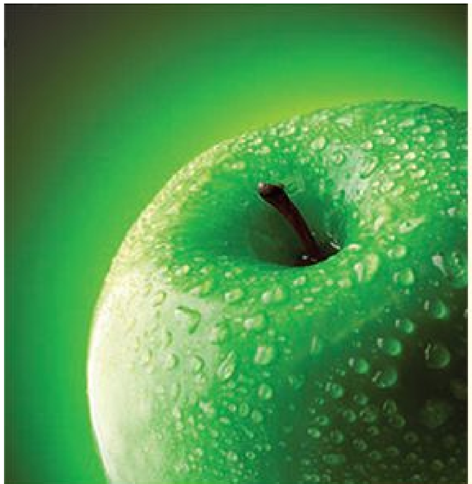


2 The Number System

SECTION

- **2.1**

Whole Numbers

- Why and when did humans invent numbers?

- Why do many mathematicians regard the invention of zero as one of the most important developments in the entire history of mathematics?

- Did you know that the base ten numeration system you use every day is called the Hindu-Arabic numeration system because it was invented by the Hindus and transmitted to the West by the Arabs? Did you know that this system has been in widespread use in the West for only 500 years?

- **A.** Use the numeral 323 to explain what “place value” means and what it means to say we have a “base ten” place value system.

- **B.** What if we had a base five system instead of a base ten system? What would the numeral 323 mean then?

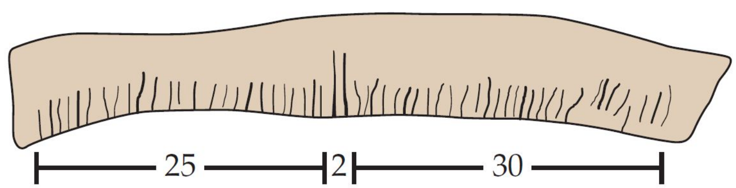
- **A.** Our system is a base ten system because we group in tens. The value of a symbol is determined by its place. In the number 323, we have the symbol “3” in two different places. The “3” on the left represents 300 while the “3” on the right represents 3 ones.

- **B.** If we used a base five system instead of a base ten system, then we would group in fives instead of tens.

Origins of Numbers and Counting

•Did you know that people had to invent counting?

•The earliest systems of counting must have been quite simple, probably tallies. The oldest archaeological evidence of such thinking is a wolf bone over 30,000 years old, discovered in the former Czechoslovakia (Figure 2.1).



•**Figure 2.1**

•On the bone are 55 notches in two rows, divided into groups of five. We can only guess what the notches represent—how many animals the hunter had killed or how many people there were in the tribe.

•Other anthropologists have discovered how shepherds were able to keep track of their sheep without using numbers to count them. Each morning as the sheep left the pen, the shepherds made a notch on a piece of wood or on some other object.

•In the evening, when the sheep returned, they would again make a notch for each sheep. Looking at the two tallies, they could quickly see whether any sheep were missing. Anthropologists also have discovered several tribes in the twentieth century that did not have any counting systems!

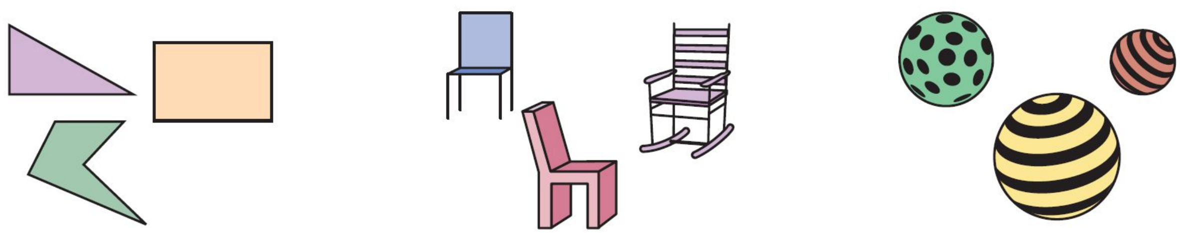
•The beginnings of what we call civilization were laid when humans made the transition from being hunter-gatherers to being farmers. Archaeologists generally agree that this transition took place almost simultaneously in many parts of the world some 10,000 to 12,000 years ago.

•It was probably during this transition that the need for more sophisticated numeration systems developed. For example, a tribe need kill only a few animals, but one crop of corn will yield many hundreds of ears of corn.

•The invention of numeration systems was not as simple as you might think. The ancient Sumerian words for one, two, and three were the words for man, woman, and many. The Aranda tribe in Australia used the word *ninta* for one and *tara* for two. Their words for three and four were *tara-ma-ninta* and *tara-ma-tara*.

•**Requirements for counting**

•In order to have a counting system, people first needed to realize that the number of objects is independent of the objects themselves. Look at Figure 2.2. What do you see?



•**Figure 2.2**

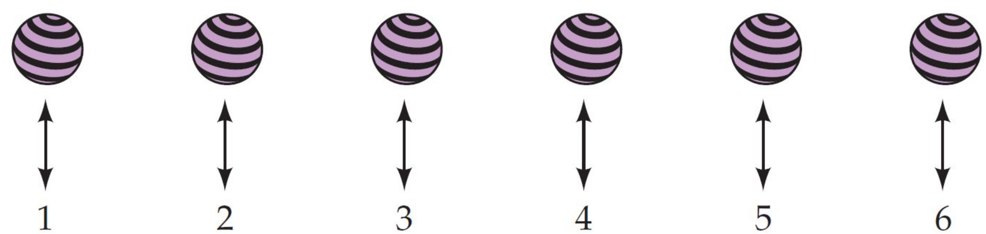
- There are three objects in each of the sets. However, the number three is an abstraction that represents an amount. Archaeologists have found that people didn’t always understand this.

•For example, the Thimshians, a tribe in British Columbia, had seven sets of words in their language for each number they knew, depending on whether the word referred to (1) animals and flat objects, (2) time and round objects, (3) humans, (4) trees and long objects, (5) canoes, (6) measures, and (7) miscellaneous objects. Whereas we would say three people, three beavers, three days, and so on, they would use a different word for “three” in each case.

- Having a counting system also requires that we recognize a one-to-one correspondence between two equivalent sets: the set of objects we are counting and the set of numbers we are using to count them.

- Many children miscount objects for some time, either counting too many or counting too few, because they do not yet realize that they need to say the next number each time they touch the next object.

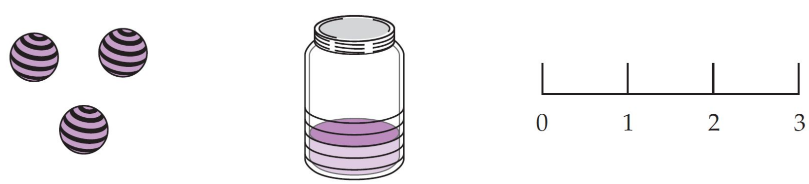
•It takes some time for them to realize that each object represents the next number, as shown in Figure 2.3—that is, that there is a one-to-one correspondence between the set of objects and the set of numbers.



•**Figure 2.3**

•There is another aspect of counting that needs to be noted. Most people think of numbers in terms of counting discrete objects. However, this is only one of the contexts in which numbers occur.

•For example, in Figure 2.4, there are 3 balls, there are 3 ounces of water in the jar, and the length of the line is 3 centimeters. In the first case, the 3 tells us how many objects we have.



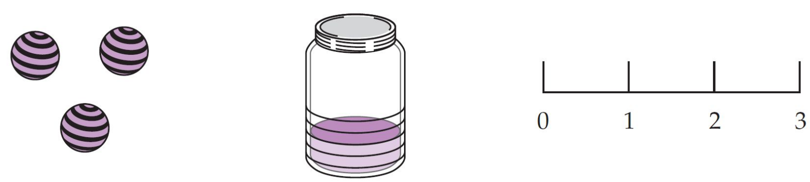
•**Figure 2.4**

- However, in the two latter cases, the number tells how many of the units we have. In this example, the units are ounces and centimeters.

- Working with numbers that represent sets of objects is more concrete than working with numbers that represent measures.

- We distinguish between **number**, which is an abstract idea that represents a quantity, and **numeral**, which refers to the symbol(s) used to designate the quantity.

•While the number of balls in Figure 2.4 is three, the numeral, or symbol, we would use to represent that number is 3. But the ancient Romans would use the numeral III to represent that number.



•**Figure 2.4**

Patterns in Counting

- As humans developed names for amounts larger than the number of fingers on one or two hands, the names for the larger amounts were often combinations of names for smaller amounts. Can you find the patterns and fill in the blanks for the three systems in Table 2.1?

Number	Inuit of Greenland	Aztec	Luo of Kenya
1	atauseq	ce	achiel
2	machdlug	ome	ariyo
3	pinasut	yey	adek
4	sisasmat	naui	angwen
5	tadlimat	maculli	abich

- **Table 2.1**

Number	Inuit of Greenland	Aztec	Luo of Kenya
6	achfineq-atauseq (other hand one)	chica-ce	ab-achiel
7	achfineq-machdlug	chic-ome	ab-ariyo
8	_ _ _ _ _ _ _ _ _ _ _ _ _	chicu-ey	_ _ _ _ _ _ _ _ _ _ _ _ _
9	achfineq-sisasmat	chic-naui	_ _ _ _ _ _ _ _ _ _ _ _ _
10	qulit (first foot)	matlacti	apar
11	achqaneq-atauseq (first foot one)	matlacti-on-ce	apar-achiel
12	_ _ _ _ _ _ _ _ _ _ _ _ _	_ _ _ _ _ _ _ _ _ _ _ _ _	apar-ariyo
13	_ _ _ _ _ _ _ _ _ _ _ _ _	_ _ _ _ _ _ _ _ _ _ _ _ _	_ _ _ _ _ _ _ _ _ _ _ _ _
15	achfechsaneq (other foot)	caxtulli	_ _
16	_ _ _ _ _ _ _ _ _ _ _ _ _	_ _ _ _ _ _ _ _ _ _ _ _ _	_ _ _ _ _ _ _ _ _ _ _ _ _
20	inuk navdlucho (a man ended)	cem-poualli	piero-ariyo

•**Table 2.1**

- What patterns did you see?

- People who have investigated the development of numeration systems, from prehistoric tallies to the Hindu-Arabic system, have discovered that most of the numeration systems had patterns, both in the symbols and in the words, around the amounts we call five and ten.

•However, a surprising number of systems also showed patterns around 2, 20, and 60. For example, the French word for *eighty, quatre-vingts*, literally means “four twenties.”

•As time went on, people developed increasingly elaborate numeration systems so that they could have words and symbols for larger and larger amounts. We will examine historical numeration systems to not only help us understand the development of place value but also to understand more deeply our own base ten numeration system.

The Egyptian Numeration System

•The earliest known written numbers are from about 5000 years ago in Egypt. The Egyptians made their paper from a water plant called papyrus that grew in the marshes.

•They found that if they cut this plant into thin strips, placed the strips very close together, placed another layer crosswise, and finally let it dry, they could write on the substance that resulted. Our word *paper* derives from their word *papyrus*.

•Symbols in the Egyptian system

•The Egyptians developed a numeration system that combined picture symbols (hieroglyphics) with tally marks to represent amounts. Table 2.2 gives the primary symbols in the Egyptian system. The Egyptians could represent numerals using combinations of these basic symbols.

1,000,000	100,000	10,000	1000	100	10	1
𓁨	𓆐	𓂭	𓆼	𓍢	𓎆	𓏺
Astonished person	Polliwog or burbot fish	Pointing finger	Lotus flower	Scroll	Heelbone	Staff, stroke

•Table 2.2

•Translate the following Egyptian numerals into our system.

•1 

•2. 

•Translate the following amounts into Egyptian numerals.

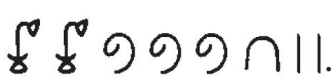
•**3.** 1202

4. 304

- **Working with the Egyptian system**
- Take a few minutes to think about the following questions.
- **1.** What do you notice about the Egyptian system? What patterns do you see?
- **2.** What similarities do you see between this and the more primitive systems we have discussed?
- **3.** What limitations or disadvantages do you find in this system?
- **4.** Is this a place value system? Why or why not?

•The Egyptian numeration system resembles many earlier counting systems in that it uses tallies and pictures. In this sense, it is called an *additive system*. One of my students called this an “image system” because it is the image that determines the value of the symbol, not the place that it is located. So, it is not a place value system.

•Look at the way this system represents the amount 2312. In one sense, the Egyptians saw this amount as 1000 + 1000 + 100 + 100 + 100 + 10 + 1 + 1 and wrote it as



•In an **additive system**, the value of a number is literally the sum of the value of the numerals (symbols).

•However, this system represents a powerful advance: The Egyptians created a new digit for every *power of ten*.

•They had a digit for the amount 1. To represent amounts between 1 and 10, they simply repeated the digit. For the amount 10, they created a new digit. All amounts between 10 and 100 can now be expressed using combinations of these two digits. For the amount 100, they created a new digit, and so on.

•These amounts for which they created digits are called **powers of ten**. You may recall, from your work with exponents from algebra, that we can express 10 as 10^1 and 1 as 10^0.

•Thus,

1,000,000	100,000	10,000	1000	100	10	1
$10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$	$10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$	$10 \cdot 10 \cdot 10 \cdot 10$	$10 \cdot 10 \cdot 10$	$10 \cdot 10$	10	1
10^6	10^5	10^4	10^3	10^2	10^1	10^0

- The Egyptian system was a remarkable achievement for its time. Egyptian rulers could represent very large numbers. One of the primary limitations of this system was that computation was extremely cumbersome.

- It was so difficult, in fact, that the few who could compute enjoyed very high status in the society.

The Roman Numeration System

- The Roman system is of historical importance because it was the numeration system used in Europe from the time of the Roman Empire until after the Renaissance. In fact, several remote areas of Europe continued to use it well into the twentieth century. Some film makers still list the copyright year of their films in Roman numerals. Where are some other places where we still use Roman numerals?

•Symbols in the Roman system

•Table 2.3 gives the primary symbols used by early Romans and later Romans.

Amount	Early Roman	Later Roman
1	I	I
5	V or Λ	V
10	X	X
50	↓	L
100	⊙	C
500	ᗡ and D	D
1000	⦶ or ∞	M

•Table 2.3

- **Working with the Roman system**
- Take a few minutes to think about the following questions.
- **1.** What do you notice about the Roman system? What patterns do you see?
- **2.** What similarities do you see between this system and the Egyptian system?
- **3.** What limitations or disadvantages do you find in this system?
- **4.** Is this a place value system? Why or why not?

- Like the Egyptians, the Romans created new digits with each power of ten, that is, 1, 10, 100, 1000, and so on. However, the Romans also created new digits at “halfway” amounts—that is, 5, 50, 500, and so on. Why do you think they did this?

- This invention reduced some of the repetitiveness that encumbered the Egyptian system. For example, 55 is not X X X X X I I I I I but LV.

•Basically, the Roman system, like the Egyptian system, was an additive system. However, the Later Roman system introduced a *subtractive* aspect. For example, Ⅳ can be seen as “one before five.”

•This invention further reduced the length of many large numbers. However, this is still not a place value system since the value of the symbol itself is not determined by where it is placed.

•As in the Egyptian system, computation in the Roman system was complicated and cumbersome, and neither system had anything resembling our zero.

The Babylonian Numeration System

- The Babylonian numeration system is a refinement of a system developed by the Sumerians several thousand years ago. Both the Sumerian and Babylonian empires were located in the region occupied by modern Iraq. The Sumerians did not have papyrus, but clay was abundant.

- Thus, they kept records by writing on clay tablets with a pointed stick called a stylus, just like we use a stylus with today's tablets. Thousands of clay tablets with their writing and numbers have survived to the present time; the earliest of these tablets were written almost 5000 years ago.

- **Symbols in the Babylonian system**
- Because the Babylonians had to make their numerals by pressing into clay instead of writing on papyrus, their symbols could not be as fancy as the Egyptian symbols.

- They had only two symbols, an upright wedge that symbolized “one” and a sideways wedge that symbolized “ten.” In fact, the Babylonian writing system is called *cuneiform*, which means “wedge-shaped.”

•You can see some pictures of these ancient tablets by searching on the web for "Babylonian mathematics tablets."

Amount	Symbol
1	
10	

•Amounts could be expressed using combinations of these numerals, for example, 23 was w.s

•However, being restricted to two digits creates a problem with large amounts. The Babylonians’ solution to this problem was to choose the amount 60 as an important number.

•Unlike the Egyptians and the Romans, they did not create a new digit for this amount. Rather, they decided that they would have a new *place*.

- For example, the amount 73 was represented as .
- That is, the  at the left represented one 60 and t
- to the right represented 13. In other words, they saw 73 as 60 + 13.

- Similarly,  was seen as six 60s plus 12, or 372.

•Translate the following Babylonian numerals into our system.

•1 

•2. 

•3. 

•Translate the following amounts into Babylonian numerals.

•**4.** 1202

5. 304

- **Working with the Babylonian system**
- Take a few minutes to think about the following questions.
- **1.** What do you notice about the Babylonian system? What patterns do you see?
- **2.** What similarities do you see between this system and the Egyptian and Roman systems?
- **3.** What limitations or disadvantages do you find in the Babylonian system?
- **4.** Is this a place value system? Why or why not?

•**Place value**

•We consider the Babylonian system to be a *place value system* because the value of a numeral depends on its place (or position) in the number. To represent larger amounts, the Babylonians invented the idea of the value of a digit being a function of its place in the numeral.

•This is the earliest occurrence of the concept of **place value** in recorded history. With this idea of place value, they could represent any amount using only two digits, 

•and .

- We are not sure why the Babylonians chose 60, although many hypotheses have been set forth. One is that because many numbers divide 60 evenly (1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60), this choice made calculations with fractions much easier.

- Their choice of 60 affects our lives today, for it was they who divided a circle into 360 parts and divided an hour into 60 minutes.

- We can understand the value of their system by examining their numerals with expanded notation. Look at the following Babylonian number:

- Because the  occurs in the first (or rightmost) place, its value is simply the sum of the values of the digits—that is, 10 + 10 + 1 = 21. However, the value of the  in the second place is determined by multiplying the face value of the digits by 60—that is, 23 · 60.

- The value of the  in the third place is determined by multiplying the face value of the digits 60^2 —that is, $2 \cdot 60^2$.

- The value of this amount is

$$(2 \cdot 60^2) + (23 \cdot 60) + 21 = 7200 + 1380 + 21 = 8601$$

- Thus, in order to understand the Babylonian system, you have to look at the face value of the digits *and* the place of the digits in the numeral. The value of a numeral is no longer determined simply by adding the values of the digits. One must take into account the place of each digit in the numeral.

- The Babylonian system is more sophisticated than the Egyptian and Roman systems. However, there were some "glitches" associated with this invention.

- What if there were nothing in a place? For example, how could the Babylonians represent the amount 3624?

- **The need for a zero**

- If we represent this amount from the Babylonian perspective, we note that

- $60^2 = 3600$. Thus, the Babylonians saw 3624 as 3600 + 24

- They would use in the third place to represent 3600, and they would in the first place to represent 24, but the second place is empty.

- Thus, if they wrote how was the reader to know that this was not 60 + 24 = 84? Again, try to imagine yourself as a Babylonian. How might you solve this problem?

- A Babylonian mathematical table from about 300 B.C. contains a new symbol that acts like a zero.

•Using this convention, they could represent 3624 as

•The slightly sideways wedges indicate that the second place is empty, and thus we can unambiguously interpret this numeral as representing

$$60^2 + 0 + 24 = 3624$$

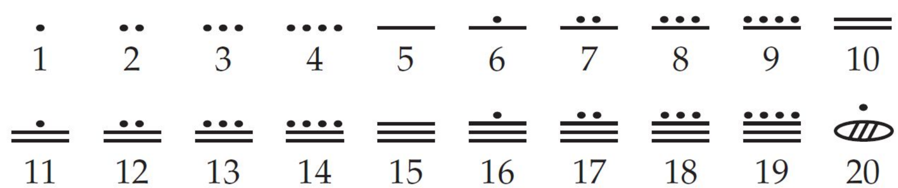
•This later Babylonian system is thus considered by many scholars to be the first place value system because the value of every symbol depends on its place in the numeral and there is a symbol to designate when a place is empty.

The Mayan Numeration System

•One of the most impressive of the ancient numeration systems comes from the Mayans, who lived in the Yucatan Peninsula in Mexico, around the fourth century A.D.

•Many mathematics historians credit the Mayans as being the first civilization to develop a numeration system with a fully functioning zero.

•The table below shows their symbols for the amounts 1 through 20. Note that they wrote their numerals vertically.



•Their numeral for 20 consisted of one dot and their symbol for zero. Thus, their numeral for 20 represents one group of 20 and 0 ones, just as our symbol for ten represents one group of 10 and 0 ones.

- Theirs was not a pure base twenty system because the value of their third place was not 20×20 but 18×20.

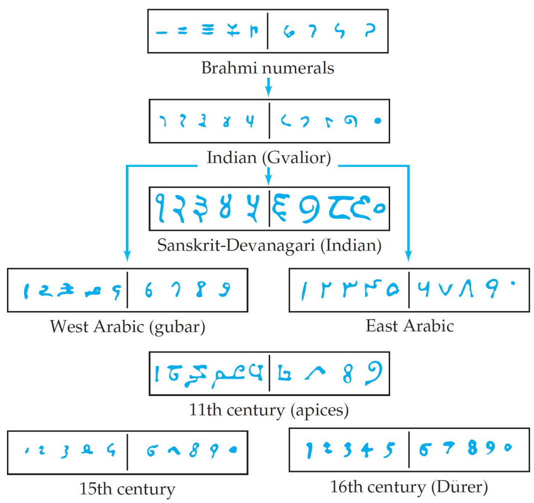
- The value of each succeeding place was 20 times the value of the previous places. The values of their first five places were 1, 20, 360, 7200, and 14,400.

The Development of Base Ten: The Hindu-Arabic System

- The numeration system that we use was developed in India around A.D. 600. By A.D. 800, news of this system came to Baghdad, which had been founded in A.D. 762. Leonardo of Pisa travelled throughout the Mediterranean and the Middle East, where he first heard of the new system.

- In his book *Liber Abaci* (translated as *Book of Computations*), published in 1202, he argued the merits of this new system. In 1229 the City Council of Florence, Italy, passed a law forbidding the use of base ten numbers when entering records of money in account books. Numbers had to be written out.

•Figure 2.5 traces the development of the ten digits that make up our numeration system.



•**Figure 2.5**

- The development of numeration systems from the most primitive (tally) to the most efficient (base ten) has taken tens of thousands of years. Although the base ten system is the one you grew up with, it is also the most abstract of the systems and possibly the most difficult initially for children.

- Stop and reflect on what you have learned thus far in your own investigations. Imagine describing this system to a Babylonian, Egyptian, or Roman who has suddenly been transported into our time.

Advantages of Base Ten

- Our **base ten** numeration system has several characteristics that make it so powerful.

- **No tallies**

- The base ten system has no vestiges of tallies. Any amount can be expressed using only 10 **digits:** 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. In fact, the word *digit* literally means “finger.”

Egyptian	Roman	Babylonian	Hindu-Arabic
𓏺	I	𒁹	1
𓏺𓏺	II	𒁹𒁹	2
𓏺𓏺𓏺	III	𒁹𒁹𒁹	3
𓏺𓏺𓏺𓏺	IV	𒁹𒁹𒁹𒁹	4
𓏺𓏺𓏺𓏺𓏺	V	𒁹𒁹𒁹𒁹𒁹	5
𓏺𓏺𓏺𓏺𓏺𓏺	VI	𒁹𒁹𒁹𒁹𒁹𒁹	6
𓏺𓏺𓏺𓏺𓏺𓏺𓏺	VII	𒁹𒁹𒁹𒁹𒁹𒁹𒁹	7
𓏺𓏺𓏺𓏺𓏺𓏺𓏺𓏺	VIII	𒁹𒁹𒁹𒁹𒁹𒁹𒁹𒁹	8
𓏺𓏺𓏺𓏺𓏺𓏺𓏺𓏺𓏺	IX	𒁹𒁹𒁹𒁹𒁹𒁹𒁹𒁹𒁹	9
𓎆	X	𒌋	10
𓎆𓎆	XX	𒌋𒌋	20
𓎆𓎆𓎆𓎆𓎆	L	𒌋𒌋𒌋𒌋𒌋	50
𓎆𓎆𓎆𓎆𓎆𓎆	LX	𒁹	60
𓍢	C	𒁹 𒌋𒌋𒌋𒌋	100

•**Table 2.4**

- **Decimal system**
- The base ten system is a **decimal** system, because it is based on groupings (powers) of ten. The value of each successive place to the left is ten times the value of the previous place:

- 100,000 10,000 1000 100 10 1
- Ten ones make one ten.
- Ten tens make one hundred.
- Ten hundreds make one thousand.
- Ten thousands make ten thousand.

•This system is consistent when we go to the right of the decimal point.

•**Expanded form**

•When we represent a number by decomposing it into the sum of the values from each place, we are using **expanded form**. There are different variations of expanded form.

- For example, all of the expressions below emphasize the structure of the numeral, 234—some more simply and some using exponents.
- $234 = 200 + 30 + 4$
- $= 2 \cdot 100 + 3 \cdot 10 + 4 \cdot 1$

$$= 2 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0$$

Note: $10^1 = 10$ and $10^0 = 1$.

- **The concept of zero**
- “The invention of zero marks one of the most important developments in the whole history of mathematics.”

- It was the genius of some person or persons in ancient India to develop this idea, which made for the most efficient system of representing amounts and also made computation much easier. In the beginning, this concept was difficult for many to accept.

•A French writer in the 1400s referred to zero as nothing but “a sign which creates confusion and difficulties.” Some writers ridiculed this new system, thinking that it would go away once people came to their senses and went back to Roman numerals: “Just as the rag doll wanted to be an eagle, the donkey a lion, and the monkey a queen, the *cifra* [zero] put on airs and pretended to be a digit.”

•One of the most difficult aspects of this system is that the symbol 0 has two related meanings: In one sense, it works just like any other digit (it can be seen as the number 0), and at the same time, it also acts as a place holder.

•It takes young school-children several years to understand this fully and accept it. So, is zero a number? Some will argue that it is “nothing” so it is not a number.

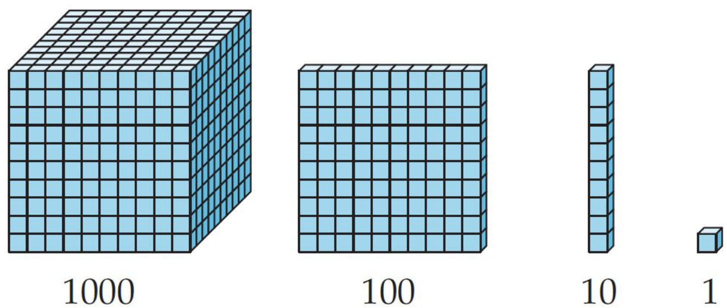
•Others say that it is just a place holder. However, zero does represent a quantity (and therefore is a number). Let’s illustrate it this way.

•If you were asked how many letter “a”s were in the word “banana,” you would answer 3.

- What if you were asked how many “e”s were in the word “banana”? You would answer zero. Zero is the number of “e”s just as three is the number of “a”s.

- We will use pictures of base ten blocks to help us understand place value and the operations more deeply.

• There are also virtual versions available on the National Library of Virtual Manipulatives website (nlvm.usu.edu). (You will find base blocks under the “number and operations” link.)



•Notice that the ones place is represented by a unit that we will call a "**single**." It takes ten of the singles to make one **long**; 10 of the longs to make one **flat**; and 10 flats to make one **large cube**.

•This visual representation also helps us to geometrically see that the 100 place value is 10^2 because it is a 10 by 10 square and that the 1000 place is 10^3 because it is a 10 by 10 by 10 cube.

- What if we had a base five system instead of a base ten system? How would these models change? Instead of grouping in tens, we would trade when there were five.

- So, five singles would equal one long (of five); five longs would equal a flat (which would be a five by five square); five flats would be a cube (which would be a cube five by five by five).

- We will explore other bases in this course for several reasons. First, it helps us to understand our base ten place value system more deeply.

•Second, it helps us to understand how difficult this concept is for young children when they are first learning it. Lastly, there are places where in essence we have to think in other bases. For example, when telling time, we are using a modified base sixty system.

•Many children, when first learning to tell time, will count time like this: 3:58, 3:59, 3:60, 3:61, … 3:98, 3:99, 4:00. Why do you think they do this?

- They are used to exchanging when they get to 100, but to go up in the digit to the left when they reach 60 is very different.

- Our coin system of pennies, nickels, and quarters is essentially a base five system.

Units

•There is an old parable that says a journey of 1000 miles begins with a single step. The same can be said for counting. We always begin with 1.

•However, unlike the phrase “a rose is a rose is a rose,” a 1 is not always the same. For example, a 1 in the millions place represents 1 million. This is the power of our numeration system, but it is very abstract.

•We will elaborate this idea of unit now because it will recur throughout this book. The first place in our system is called the ones place, the singles place, or the units place.

•By the time children come to school, they are generally very comfortable with the idea of counting one at a time. Over time they can count higher and higher.

•When counting objects, one is our key term. When asked to count a pile of objects, for example, 240 pennies, children will count one at a time.

•However, if they lose their count, they have to start all over. Some children realize that they can put the pennies into piles of 10.

•Now if they lose count, they can go back and count by tens, for example, 10, 20, 30, 40, and so on.

•In this case, 10 is a **simple unit**, that is, it is composed of a number of smaller units. Some children can see that 1 pile is also 10 pennies.

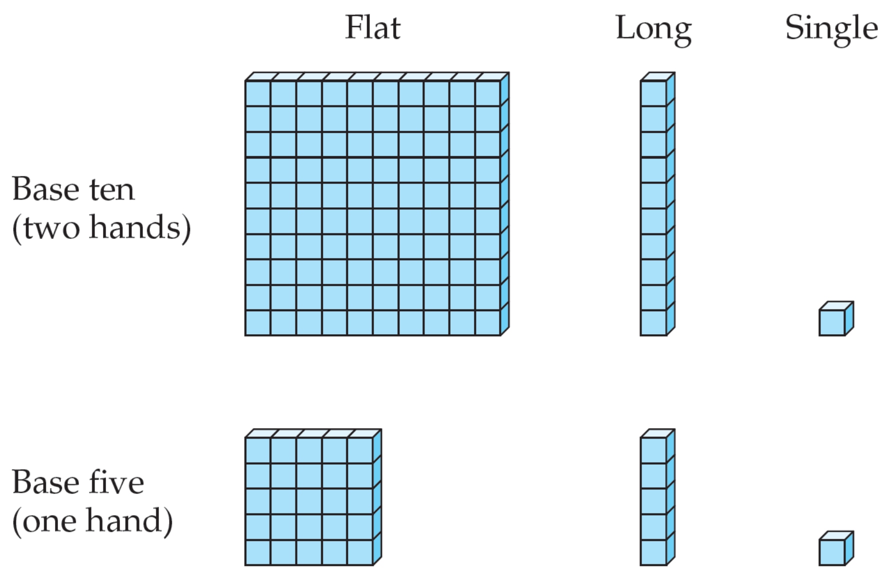
•To be able to hold these two amounts simultaneously is a challenge for young children, and it is an essential milestone.

- We have **composite units** everywhere: 100 is equivalent to ten 10s, 1000 is equivalent to ten 100s. In fact, our language shows this: some people will say thirty-four hundred for 3400 instead of three thousand four hundred.

- We talk about 1 dozen eggs, a case of soda (24 cans), and a pound (16 ounces). When we say that we will need 6 dozen eggs for a pancake breakfast fundraiser, we can see 6 dozen, and we also know that this is 72 individual eggs.

- The people who developed base ten decided to base it on two hands. What if they had decided to base it on one hand? That is, what if one-zero had come not after we counted two hands but after we counted one hand? Our counting would look like this: 1, 2, 3, 4, 10, ...

•The manipulatives (see the figure below) would have the same basic shape as the base ten blocks.



•Because the structures of this new base are the same as in the system you grew up with, the counting follows the same rules. The table below shows the beginnings of the new system.

1	2	3	4	10
11	12	13	14	20
21	22	23	24	30
31	32	33	34	40
41	42	43	44	?

•If you find yourself struggling, make your own set of manipulatives (cut from graph paper) and represent each number manipulatively: 1 single, 2 singles, 3 singles, 4 singles….

- On the virtual base blocks on the National Library of Virtual Manipulatives website, you can set it to base five, and then it groups the singles in fives to make a long.

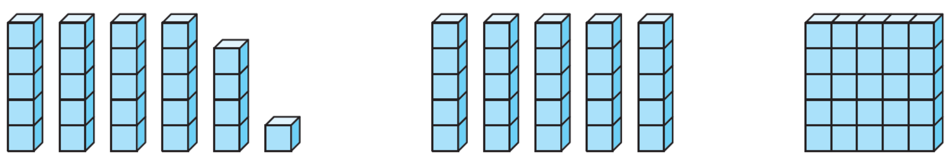
- The next number represents one hand and will now be called one-zero because this is what a long is in this system. The system continues all the way to 44. What comes next? Think carefully.

- First of all, counting simply means adding 1 each time, so the next number after 44 is 44 + 1. In *this* base, a place is full after 4. Thus, the ones place is full, and we “move” the five ones into the next place because we can trade five singles for one long.

- However, the longs place is also full once we get one more long. Thus, we trade (regroup) 5 longs for 1 flat, and we now have 1 flat, 0 longs, and 0 singles. That is, 100_{five} is the next number after 44_{five}.

•If you don’t see this, use manipulatives. In the first figure at the left, we see 44 (4 longs and 4 singles) plus 1. In the second figure, we see those five singles have been traded for a long.

•In the third figure, we see the five longs have been traded for a flat. We now have 1 flat, 0 longs, and 0 singles—that is, 100_{five}.



- Jerome Bruner wrote that children learn mathematical ideas best if they begin at the concrete level and then move at their own rate to the symbolic level. I find this true of most adults. The most basic level is the level of manipulatives.

- If your instructor does not have base five blocks, you can make your own set by cutting flats, longs, and singles from graph paper. Many students find that actually making and moving the manipulatives helps them grasp the ideas more readily.

•At some point, many students find that they no longer need the manipulatives but that jumping all the way to using just numbers is abstract. Bruner knew this and articulated a middle level that is a pictorial level.

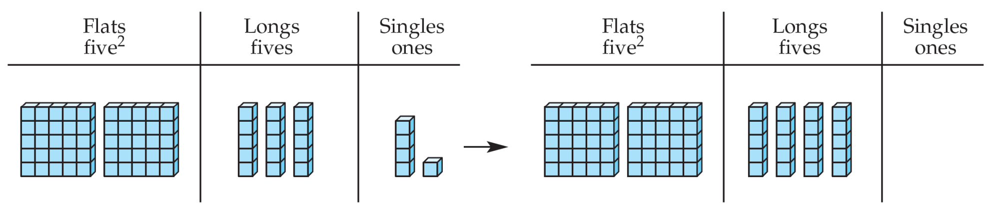
•At this level, you don't need the manipulatives but find pictures of them useful. I have come to believe that most adults learn new mathematics concepts more deeply if they first experience these concepts at the concrete level and proceed from the concrete level to the visual level and finally to the symbolic level *at their own pace* and by connecting the representations at each level.

- Explore the following using manipulatives, pictures, or virtual manipulatives.

- **A.** What number comes after 234_{five}?
- **B.** What number comes after 1024_{five}?
- **C.** What number comes before 210_{five}?
- **D.** What number comes before 3040_{five}?

- **A.** At the visual level, you can see that the next number means 2 flats, 3 longs, 4 singles, and 1 more single. Thus, we now have 2 flats, 4 longs, and 0 singles—that is, 240_{five}.

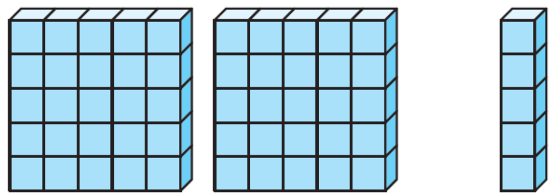
- At the symbolic level, you might reason the first problem like this: After 234_{five}, we have filled the ones place, so the ones place will now be 0, the longs place will have one more, so it will be 4, and the flats place will still be 2.



- **B.** To determine what number comes after 1024_{five}, stop and reflect for a moment. The manipulatives are like training wheels. They are useful to help the ideas develop, but eventually they need to come off, especially as the numbers get larger.

- If we add to 1024_{five}, the ones place is full—five ones becomes one more long (since five ones = 10_{five}). Now we have 3 longs. No other places are affected, so the next number is 1030_{five}.

- **C.** What number comes before 210? The answer comes more visually from the picture. Do you see how?



•The number before 210_{five} is 1 less than 210_{five}. You can see this by trading (regrouping) the long into five singles and taking one single away so that we now have 2 flats, 0 longs, and 4 singles—that is, 204_{five}.

- Another way to see it is to cover up one of the singles on the long above and realize that this represents the breaking up of that long into singles.

- **D.** What comes before 3040_{five}? You could represent this as 3 big cubes, 0 flats, 4 longs, and 0 singles. In order to take 1 away from 3040_{five}, you would have to break apart one long into five singles and take away one single.

- You now still have 3 big cubes and still have 0 flats, but now you have 3 longs and 4 singles. Thus, the number before 3040_{five} is 3034_{five}.

•At a more abstract level, as you internalize the properties of a base system, you simply know that the number before 3040_{five} must have a 4 in the ones place (just as you “know” that the number before 7090 in base ten must end in a 9).

•Since you are going backwards, you know that you have one fewer long. You also “know” that the flats and next place are not affected in this case, and thus the answer is 3034_{five}.

- Either in explorations or in class, your instructor may have you work with other bases so that your understanding of the fundamental ideas of the base and place value become deeper and deeper.

- Having explored other bases, let us now revisit the fundamental ideas of base ten. Any base—base two, base five, base ten, base twelve, whatever—will have the following fundamental characteristics.

- **1.** Any base has the same number of symbols as the number of base. In a base ten system, we have ten symbols (0–9), and in a base five system, we have five symbols (0–4). With those symbols, we can represent any amount.

- **2.** The value of each place is the base times the previous place. In base ten, the value of the places is ones, tens,
- $\text{ten}^2, \text{ten}^3,$ and so on. Similarly, in base five the value of the places is ones, fiv $\text{five}^2, \text{five}^3,$ • and so on.

- **3.** Each place can contain only one symbol. When a place is full, we “move” to the next place by trading (regrouping) to the next higher place.
- **4.** The value of a digit depends on its place in the numeral.
- **5.** The value of a numeral is determined by multiplying each digit by its place value and then adding these products.
- **6.** Zero represents an empty place and 0 represents an actual amount, with a place on the number line.

- Computers don't have fingers to count with; they just have on and off. Thus, computers begin with base two. For a variety of reasons, computers actually compute in base sixteen. Because we do not trade for a long until we have sixteen ones, how do we represent ten, eleven, up to fifteen ones? 10_{sixteen} means one long of 16, so we cannot use 10 to mean ten.

- Thus, we have to make up new digits in base sixteen for the base ten amounts between ten and fifteen. How do you think this problem was solved?

•The solution was to use the alphabet. The table below shows the numerals for 1 through 16 in base ten and base sixteen.

Base ten	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Base sixteen	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f	10

•How would you represent the following base ten numerals in base sixteen?

•**A.** 25 **B.** 100

- **A.** If we take 25_{ten} and repackage it in terms of sixteens, we have $1 \times 16 + 9$. Thus, 25 in base ten is equivalent to 19 in base sixteen. Symbolically, we can write
- $25_{ten} = 19_{sixteen}$.

- **B.** If we take 100_{ten} and repackage it in terms of sixteens, we have $100 = 6 \times 16 + 4$. Symbolically, $100_{ten} = 64_{sixteen}$.

- Our modern society deals with large numbers all the time.
  - Politicians talk about a war costing $100 billion a year.
  - The federal deficit is more than $20 trillion at the writing of this book.
  - The closest star to us is about 24,600,000,000,000 miles from earth.
  - The cleanup from Hurricane Katrina involved the removal of 500 million cubic yards of debris.
  - More than 43 million people in the United States live in poverty.

- Let’s investigate an example commonly used in elementary classrooms. If a large paper clip is about two inches long, how long would a chain of 1 million of those paper clips be?

•Well, it would be 2 million inches, but can you “feel” 2 million inches? I can’t. If we divide 2 million by 12 (the number of inches in a foot), we get 166,667 feet. That’s still not a number most people can sense.

•However, if we divide that number by 5280 (the number of feet in a mile), we get about 32 miles. Most people have a sense of 32 miles.

- Now how long would a chain of 1 billion such paper clips be?
- It would be about 32,000 miles, because a billion is a thousand million.
- How long would a chain of 1 trillion paper clips be?

- 32,000,000 miles. The distance around the earth is about 24,000 miles. The moon is about $\frac{1}{4}$ million miles from earth, so one round trip would be $\frac{1}{2}$ million and two round trips would be 1 million, so 64 round trips would be 32 million miles.

•1 million paper clips	• 32 miles	•A bit longer than a marathon
•1 billion paper clips	• 32,000 miles	•More than the distance around the earth
•1 trillion paper clips	•32,000,000 miles	•64 round trips to the moon

- Were you surprised at how much bigger a billion is than a million; how much bigger a trillion is than a billion? Most people are. When we are not able to have a reference for thinking about large amounts, they literally swim in our heads. They lose their reality.

- If we could really sense $20 trillion, we would clamor to reduce the debt. If we could really sense the number 43 million people living in poverty in the United States, then we would likely take action.