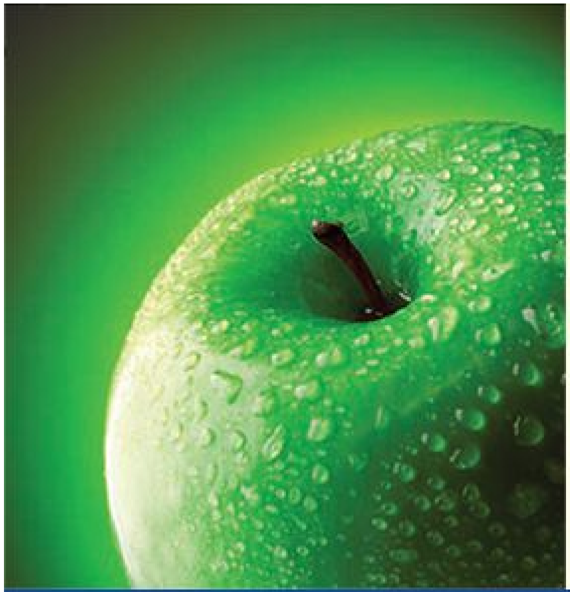


4 Understanding Multiplication and Division

SECTION

4.3

Understanding Multiplication and Division of Fractions

What do you think?

- Why do we not need a common denominator when multiplying or dividing fractions?

- If whole number multiplication means to “make bigger,” why does fraction multiplication sometimes “make smaller”?

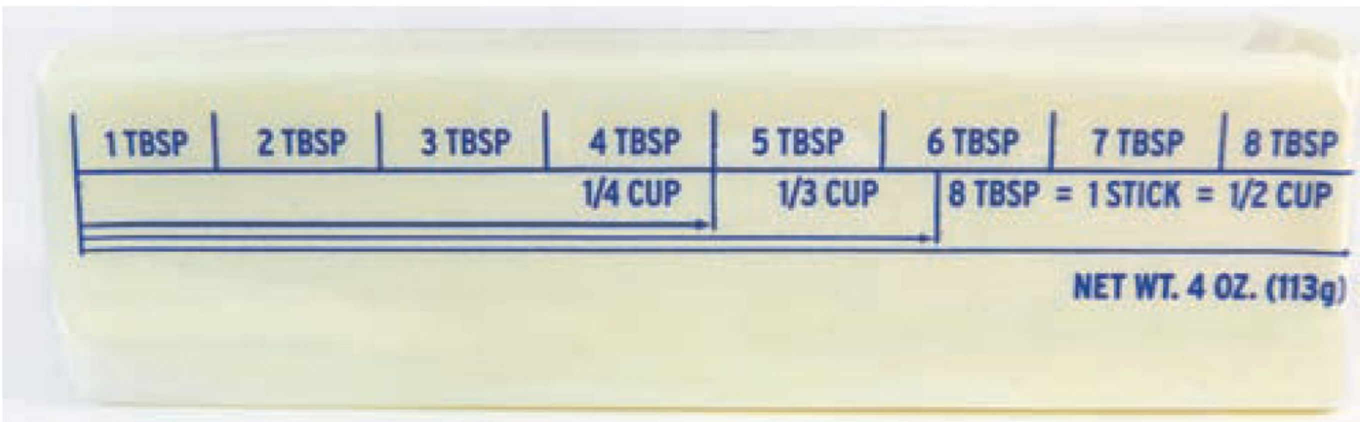
- How is multiplication and division of fractions connected to multiplication of whole numbers?

Multiplication with Fractions

Multiplication with Fractions

Concrete and contextual

If a recipe calls for $\frac{1}{2}$ of a cup of butter, how much butter is needed to make $\frac{1}{2}$ of the recipe?



Multiplication with Fractions

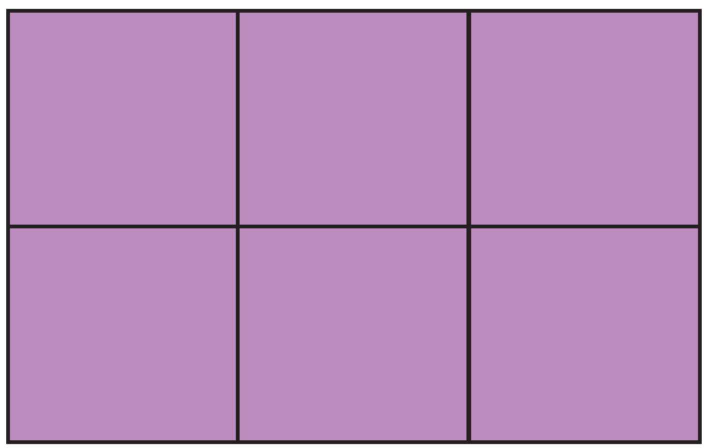
Just like when we talked about 2 × 3 being 2 groups of 3, here we want $\frac{1}{2}$ of a group of $\frac{1}{2}$ of a cup of butter, so this is $\frac{1}{2} \times \frac{1}{2}$.

The markings on this stick of butter help us to see that $\frac{1}{2}$ of $\frac{1}{2}$ of a cup of butter would equal $\frac{1}{4}$ of a stick of butter.

Multiplication with Fractions

Concrete rectangular array

Remember in the previous section we looked at the rectangular array model of multiplication where, for example, 2 × 3 would be modelled with a rectangle that is 2 by 3.

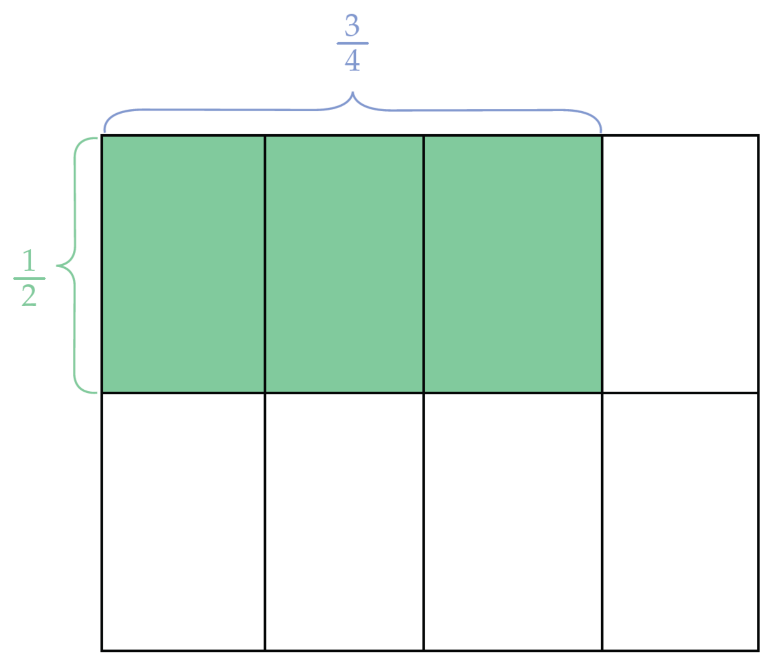


Multiplication with Fractions

Similarly, $\frac{1}{2} \times \frac{3}{4}$ can be thought of as a rectangle $\frac{1}{2}$ by $\frac{3}{4}$, or taking $\frac{1}{2}$ of $\frac{3}{4}$. Take a piece of paper, and fold it horizontally in half to show $\frac{1}{2}$. Then fold it vertically into fourths to show the $\frac{3}{4}$.

The answer will be the part of the paper where the $\frac{1}{2}$ and $\frac{3}{4}$ overlap since we are taking $\frac{1}{2}$ of the $\frac{3}{4}$.

Multiplication with Fractions



The answer is $\frac{3}{8}$ since there are 3 pieces out of 8.

Investigation 4.3a – *Understanding Multiplication of Fractions*

For each of the following problems, represent the situation with a diagram and determine the answer from the diagram. As you are doing this, think back to the models we developed for whole-number multiplication.

Which problems connect well to repeated addition? To a rectangular array? Which problems do not connect well? Why don’t they?

Which model (area, linear, or set) best represents each problem?

A. Julio runs 4 times a week. His route is $2\frac{1}{4}$ miles long. miles long. How many miles does he run each week?

Investigation 4.3a – *Discussion*

This connects nicely to multiplication as repeated addition: 4 times $2\frac{1}{4}$.

We can represent this problem on a number line, as in Figure 4.16.

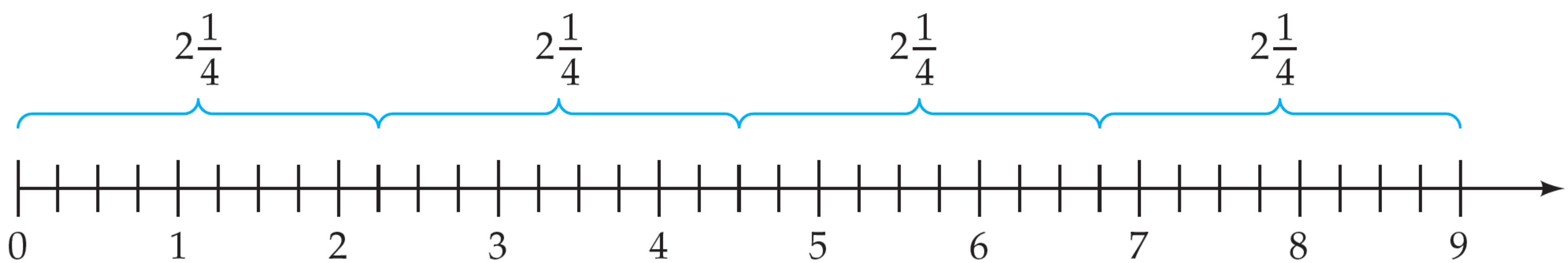


Figure 4.16

Investigation 4.3a – *Understanding Multiplication of Fractions*

B. A group of investors purchased a rectangular parcel of land that is $\frac{3}{4}$ of a mile long and $\frac{2}{3}$ of a mile wide.

How many square miles did they buy?

Investigation 4.3a – *Discussion*

This problem is definitely not repeated addition, but it does connect to the area model of multiplication and can be represented nicely with an area model of fractions. Here we can say that we want $\frac{2}{3}$ of $\frac{3}{4}$ (of 1 square mile).

If the large square in Figure 4.17 represents 1 mile by 1 mile, the shaded region represents the area of the land that the investors bought: $\frac{3}{4}$ of a mile by $\frac{2}{3}$ of a mile.

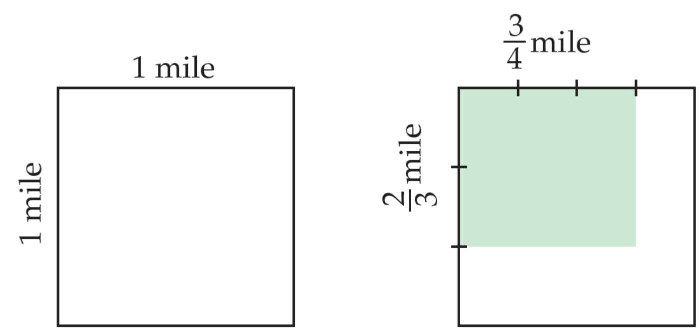


Figure 4.17

Investigation 4.3a – *Discussion* continued

But how do we give a name to this amount? Try to do this yourself.

Just as we decomposed and recomposed numbers in working with whole numbers, we need to decompose this (shaded) amount in such a way that we can recompose the amount as a set of equal-size pieces.

We can create equal-size pieces (needed to name a fraction) by dividing the square first into fourths (with vertical lines) and then into thirds (with horizontal lines).

Investigation 4.3a – *Discussion* continued

If we put the two diagrams together and shade in the area enclosed by $\frac{3}{4}$ times $\frac{2}{3}$, we have the figure on the right in Figure 4.18.

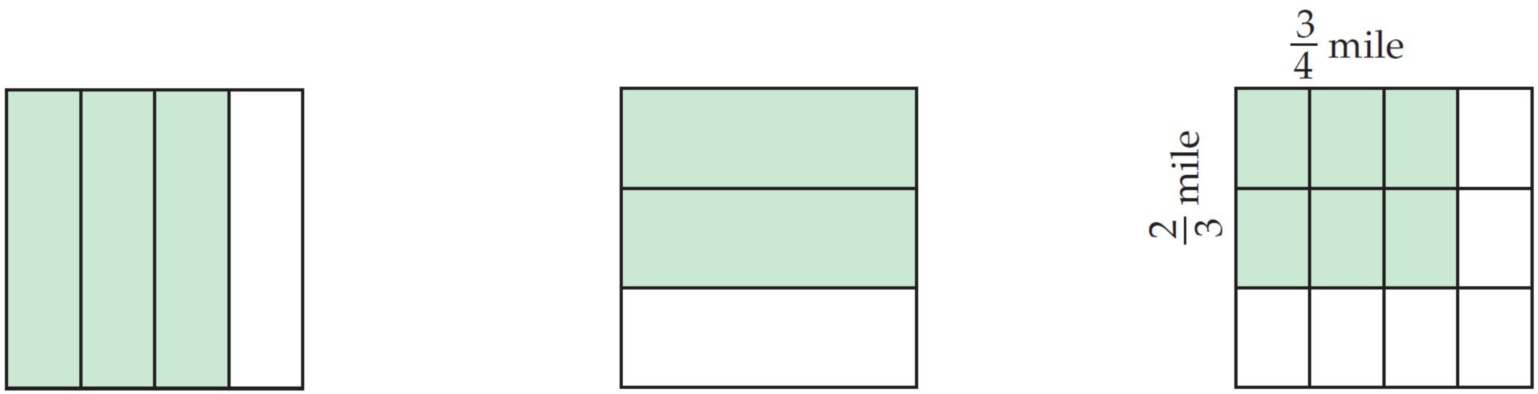


Figure 4.18

Investigation 4.3a – *Discussion* continued

How can you determine the answer now from the diagram? How can you justify that answer?

Because our unit (1 square mile) has been divided into 12 equal-size regions, each region has a value of $\frac{1}{12}$ square mile.

The plot of land covers 6 of these rectangles, so its value is $\frac{6}{12}$ or $\frac{1}{2}$ square mile.

Investigation 4.3a – *Discussion* continued

This model helps to explain the procedure of “multiply straight across” that is often used for multiplying fractions.

The shaded rectangle is 2 by 3 because of the numerators. The entire unit is 3 by 4 because of the denominators.

There is a difference between multiplying whole numbers and multiplying fractions that bears noting, for it often baffles children when they encounter fraction computations for the first time.

Investigation 4.3a – *Discussion* continued

One notion of multiplication that many people (often unconsciously) get from working with whole numbers is that “multiplication makes bigger and division makes smaller,” but just the opposite is true when we multiply proper fractions!

Investigation 4.3a – *Understanding Multiplication of Fractions*

C. If $\frac{2}{3}$ of the class went on a field trip and there are 24 students in the class, how many students went on the field trip?

Investigation 4.3a – *Discussion* continued

The operator context can be thought of as a stretching or a shrinking process. In this case, we have a whole (24 students), and we are shrinking that whole—that is, we are considering a part $\left(\frac{2}{3}\right)$ of that whole.

Figure 4.19(a) represents the problem using a set model, with each dot representing one person.

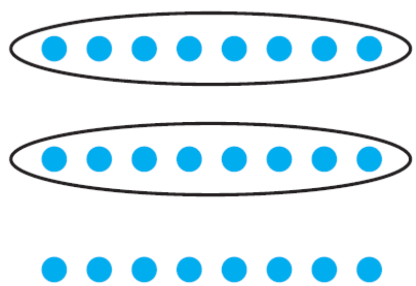


Figure 4.19(a)

Investigation 4.3a – *Discussion* continued

Because we have $\frac{2}{3}$ of the people, the $\frac{2}{3}$ tells us that we have to partition this amount into 3 equal groups and then take 2 of those groups. Figure 4.19(b) represents the problem using a bar model. The whole rectangle represents the whole class.

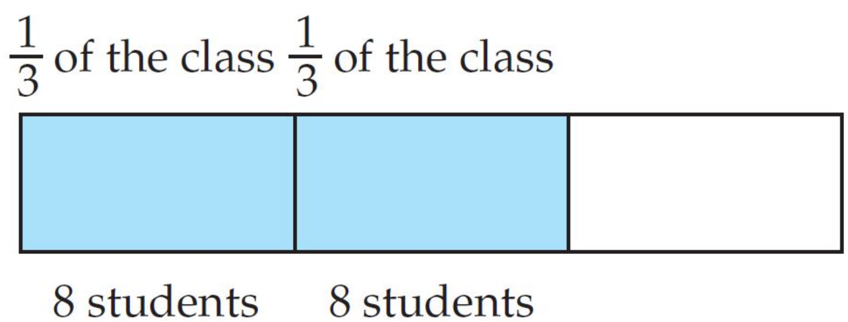


Figure 4.19(b)

Investigation 4.3a – *Discussion* continued

We have shaded in $\frac{2}{3}$ of the rectangle. If the whole class represents 24 students, then when we divide the class into 3 equal regions, each region must represent 8 students (that is, $8 + 8 + 8 = 24$ or $3 \cdot 8 = 24$).

Both models quickly produce the correct answer of 16 students.

Connecting Multiplication of Fractions to the Multiplication Algorithm

Connecting Multiplication of Fractions to the Multiplication Algorithm

As we did with multiplication of whole numbers, we will develop the algorithm with the rectangular model. Let us do so with the problem

$$\frac{3}{4} \cdot 2\frac{1}{2}$$

Let us first represent the problem with a diagram (Figure 4.20).

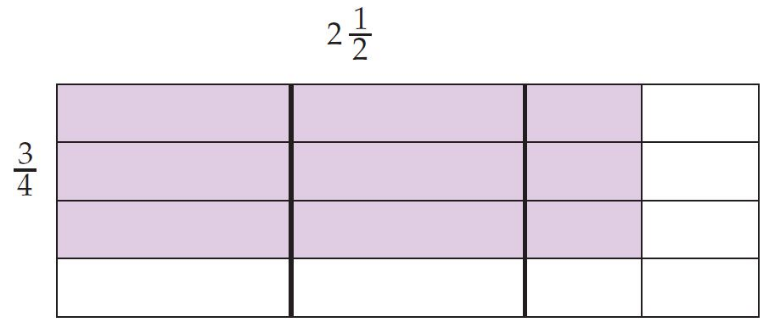


Figure 4.20

Connecting Multiplication of Fractions to the Multiplication Algorithm

We can readily see (from Figure 4.21) that the numerator of our product must be 15 (that is, there are 15 parts). But what is the denominator?

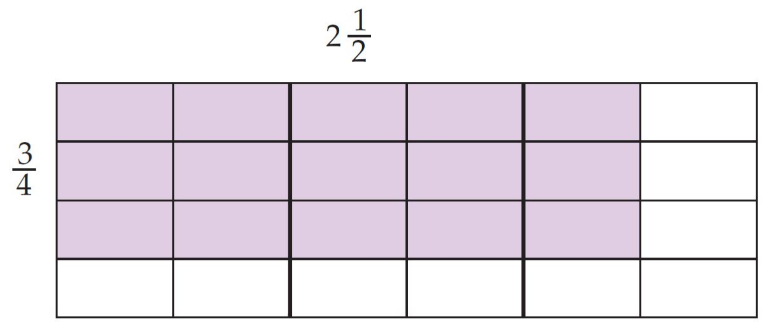


Figure 4.21

Connecting Multiplication of Fractions to the Multiplication Algorithm

The correct answer is 8. Do you see why? *We know that the value of the denominator is determined by its relationship to the unit*. That is, the denominator shows how many pieces it takes to have a value of 1, rather than how many pieces are in the whole.

Therefore, we have determined that

$$\frac{3}{4} \cdot 2\frac{1}{2} = \frac{15}{8} = 1\frac{7}{8}$$

Connecting Multiplication of Fractions to the Multiplication Algorithm

Now let us examine the algorithm and look for connections between the algorithm and the concepts.

We must first convert $2\frac{1}{2}$ to an improper fraction, and then we simply find the product of the numerators and the product of the denominators:

$$\frac{3}{4} \cdot 2\frac{1}{2} = \frac{3}{4} \cdot \frac{5}{2} = \frac{3 \cdot 5}{4 \cdot 2} = \frac{15}{8}$$

Connecting Multiplication of Fractions to the Multiplication Algorithm

If we look at the shaded region in Figure 4.21, we have 3 rows, each containing 5 regions; that is, we have 3 times 5 regions. Just as we found that the whole-number multiplication algorithm automatically regrouped for us, so too the fraction multiplication algorithm automatically creates equal-size pieces. Similarly, we have 4 · 2 pieces in each unit.

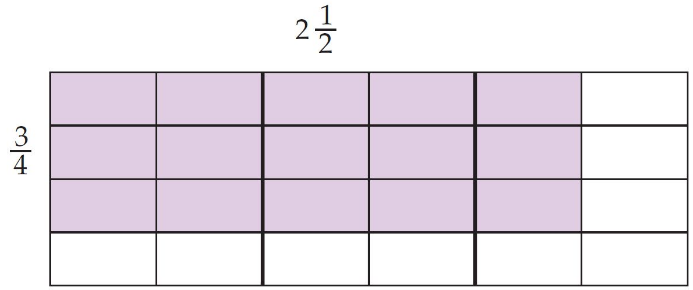


Figure 4.21

Division of Fractions

Division of Fractions

Concrete and contextual

We have 3 cups of milk and the recipe calls for $\frac{1}{2}$ cup of milk. How many batches of our recipe can be made? One way to look at division, for example 12 divided by 3, is to ask how many groups of 3 we have in 12.

Similarly, with 3 divided by $\frac{1}{2}$, we are asking how many halves are in 3. Since there are 2 halves in each 1 cup, there are 6 halves in 3 cups. So we can make 6 batches of this recipe with 3 cups of milk.

Notice how $3 \div \frac{1}{2} = 3 \times 2 = 6.$

Investigation 4.3b – *Division of Fractions*

As with most concepts being considered in this course, you probably remember the procedure. This recalls a famous saying of unknown origin: “Ours is not to reason why, just invert and multiply.”

$$\frac{3}{4} \div \frac{5}{6} = \frac{3}{4} \times \frac{6}{5} = \frac{18}{20} = \frac{9}{10}$$

Now, though, ours *is* to reason why. Let us begin our investigation of the fraction division algorithm by first examining fraction division problems in context.

Investigation 4.3b – *Division of Fractions* continued

We know the two contexts for division with whole numbers: partitioning and repeated subtraction.

Make up a story for each of the two division problems below.

First, solve the problem using what you know about division and fractions. Then respond to the following questions: Which of your stories were from the partitioning model, and which were from the repeated subtraction model? If you made up a story connected to one model, could you make up another story for the same problem using the other model?

Investigation 4.3b – *Division of Fractions* continued

A. The divisor is a whole number: $3 \div 4$

B. The divisor is a proper fraction: $3 \div \frac{2}{5}$

Investigation 4.3b – *Discussion*

A. Most people make up a story for 3 ÷ 4 using the partitioning model, and most of the stories involve sharing.

Let us consider one such story and examine a solution: Basil has 3 pints of ice cream that he wants to share equally with 3 friends—that is, among 4 people.

How much is each person’s share?

Investigation 4.3b – *Discussion* continued

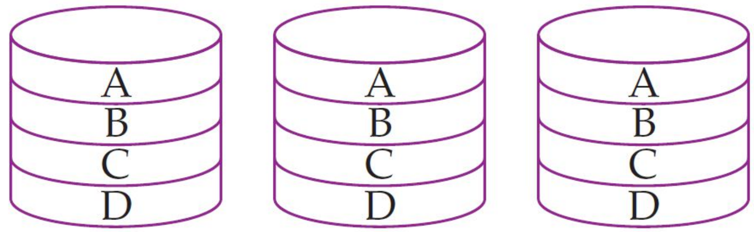


Figure 4.22

Figure 4.22 shows a partitioning (dealing) solution. That is, if we divide each pint into 4 equal parts, each person gets 1 part per pint.

Each person's share thus consists of 3 parts, each of which is $\frac{1}{4}$ pint. Therefore, each person's share is $\frac{3}{4}$ of a pint.

Investigation 4.3b – *Discussion* continued

Some people literally interpret this problem as a multiplication problem. If we are dividing the ice cream among 4 people, then each person gets $\frac{1}{4}$ of the ice cream.

This connection between the two operations produces the following equality: $3 \div 4 = 3 \cdot \frac{1}{4}$, which we just mentioned.

We will return to this relationship between multiplication and division shortly. However, first let us examine a story for the other problem.

Investigation 4.3b – *Discussion* continued

B. One story for $3 \div \frac{2}{5}$: Jake is stranded in the middle of the desert. He has 3 quarts of water, and he figures that he will drink $\frac{2}{5}$ of a quart each day. How many days' supply does he have?

Solve the problem using only your knowledge of fractions and your knowledge of division. Try drawing a model to help you understand.

This is a repeated subtraction problem because we have an amount (3 quarts) and we are specifying the size of the group ($\frac{2}{5}$ of a quart) to be repeatedly subtracted. The answer will be the number of groups we have.

Investigation 4.3b – *Discussion* continued

We have chosen to represent this problem with a number line, although an area or set model could also have been used. Each day Jake uses $\frac{2}{5}$ of a quart.

We see from Figure 4.23 that he is able to drink $\frac{2}{5}$ of a quart each day for 7 days.

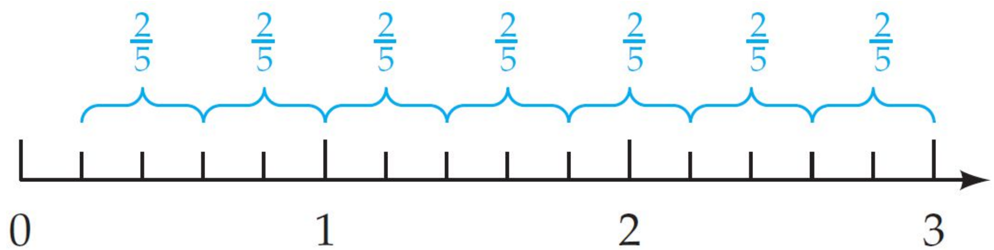


Figure 4.23

Investigation 4.3b – *Discussion* continued

However, we have a small problem: We have repeatedly subtracted $\frac{2}{5}$ of a quart, but we have $\frac{1}{5}$ of a quart left over. Does this answer contradict the answer that comes from using the algorithm, which tells us that $3 \div \frac{2}{5} = 7\frac{1}{2}$?

How do we reconcile the difference between the answer from the diagram and that from the algorithm?

It turns out that both answers are correct—they represent different ways of describing the remainder.

Investigation 4.3b – *Discussion* continued

The $\frac{1}{5}$ tells us that the remainder is $\frac{1}{5}$ of a quart; thus, we can say that at the end of 7 days, he will have $\frac{1}{5}$ of a quart left. The $7\frac{1}{2}$ tells us that the remainder is equivalent to $\frac{1}{2}$ day's water. We can think of this problem as how many $\frac{2}{5}$s are in 3?

Figure 4.23 shows us that there are $2\frac{1}{2}$ (or $\frac{5}{2}$) two-fifths in each 1 quart.

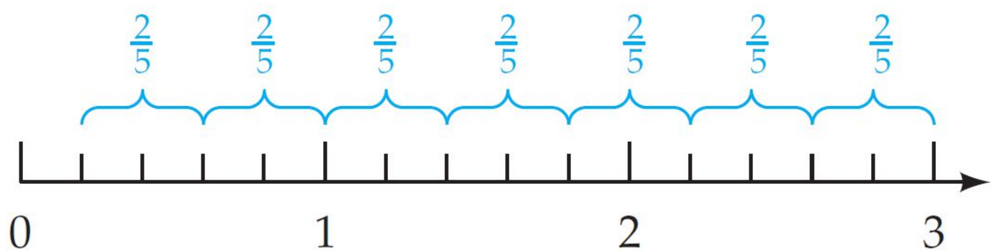


Figure 4.23

Investigation 4.3b – *Discussion* continued

Therefore, pictorially when we ask how many two-fifths portions are in 3, we can see it is 3 times $\frac{5}{2}$ portions per each quart. In other words, 3 divided by $\frac{2}{5}$ is the same as 3 times $\frac{5}{2}$.

Making Sense of the Standard Division Algorithm

Making Sense of the Standard Division Algorithm

We need to introduce a new concept to do so. Just as we discovered that every integer has an additive inverse, every nonzero fraction has a **multiplicative inverse**. These parallel concepts are shown side by side below:

-3 is the additive inverse of 3	$\frac{1}{3}$ is the multiplicative inverse of 3
$3+(-3)=0$	$\frac{1}{3}\cdot 3=1$
$-a$ is the additive inverse of a	$\frac{1}{a}$ is the multiplicative inverse of a, $a\neq 0$
$a+(-a)=0$ for any integer a	$a\cdot\frac{1}{a}=1$ for any fraction, $a\neq 0$

Making Sense of the Standard Division Algorithm

Let us now we know the way in which we defined integer subtraction in terms of addition, and then we will define fraction division in terms of fraction multiplication.

$6-3=6+(-3)$	$6 \div 3 = 6 \cdot \frac{1}{3}$
$a-b=a+(-b)$	$a \div b = a \cdot \frac{1}{b}$
That is, to subtract b, we add the (additive) inverse of b.	That is, to divide by b, we multiply by the (multiplicative) inverse of b.
In other words, one way to interpret *subtraction* is to *add* the *additive inverse* of the second number.	In other words, one way to interpret *division* is to *multiply* by the *multiplicative inverse* of the second number.

Making Sense of the Standard Division Algorithm

Thus,

$$a \div \frac{x}{y} = a \cdot \frac{y}{x}$$

Investigation 4.3c – *Estimating Products and Quotients*

In each of the following problems: First obtain a rough estimate (5 to 10 seconds), and then try to get as close as you can to the actual answer—either a refined estimate or, in some cases, an exact answer (computed mentally).

A. Anastasia walks at a pace of $3\frac{1}{2}$ miles per hour for 2 hours and 15 minutes. How far does she walk during this time?

Investigation 4.3c – *Discussion*

Strategy 1: Bound the answer

One kind of estimating involves bounding the answer. For example, the answer will be at least 6. We get this by focusing only on the whole numbers. We call 6 a lower bound.

An upper bound can be obtained by rounding both numbers to the next higher whole number. The upper bound is thus $4 \cdot 3 = 12$.

Investigation 4.3c – *Discussion* continued

Strategy 2: Get rid of one of the proper fractions (that is, rounding)

Another method of making a rough estimate involves getting rid of one of the proper fractions. We can more easily determine $3\frac{1}{2} \cdot 2 = 7$ or $3 \cdot 2\frac{1}{4} = 6\frac{3}{4}$.

Investigation 4.3c – *Discussion* continued

Strategy 3: Use partial products

We can estimate and round the four partial products:

$$3\frac{1}{2}\times 2\frac{1}{4}=\left(3+\frac{1}{2}\right)\times\left(2+\frac{1}{4}\right) = (3\times 2) + \left(3\times\frac{1}{4}\right) + \left(2\times\frac{1}{2}\right) + \left(\frac{1}{2}\times\frac{1}{4}\right)$$

$$\approx 6 + 1 + 1 + 0$$

$$\approx 8$$

Investigation 4.3c – *Estimating Products and Quotients*

B. The Jones family finds that they spend about $\frac{2}{3}$ of their income on food and rent. If their monthly income is $1635, how much is their monthly food and rent bill?

Investigation 4.3c – *Discussion*

Rough estimate (using compatible numbers):

$$\frac{2}{3} \times 1500 = 1000$$

More refined estimate (using compatible numbers and expanded form):

$$\frac{2}{3} \times 1650 = \frac{2}{3}(1500 + 150)$$

$$= 1000 + 100$$

$$= 1100$$

Investigation 4.3c – *Estimating Products and Quotients*

C. Marvin has 23 yards of cloth with which to make costumes for the play. Each costume requires $3\frac{1}{4}$ yards of material. How many costumes can he make?

Investigation 4.3c – *Discussion*

Rough estimate (using bounding):

$$23 \div 3 = 7\frac{2}{3}$$

$$23 \div 4 = 5\frac{3}{4}$$

The estimate is about 6 or 7 costumes.

Refined estimate (using repeated addition):

$$3\frac{1}{4} \times 2 = 6\frac{1}{2}, \text{ so}$$

Investigation 4.3c – *Discussion* continued

$$3\frac{1}{4} \times 4 = 13, \text{ so}$$

$$3\frac{1}{4} \times 8 = 26$$

Because 26 − 3 = 23, this estimate yields 7 costumes.

Investigation 4.3c – *Estimating Products and Quotients*

D. Shelly has 25 pounds of dog food, and each day she feeds her dog $\frac{3}{4}$ pound. How many days' worth of food does she have?

Investigation 4.3c – *Discussion*

Rough estimate (using bounding):

$$25 \div 1 = 25 \qquad \text{and} \qquad 25 \div \frac{1}{2} = 50$$

Thus, she has enough for more than 25 days and less than 50 days.

Refined estimate:

$$25 \div \frac{3}{4} = 25 \times \frac{4}{3} = \frac{100}{3} = 33\frac{1}{3}$$

which is the actual quotient. Thus, she has 33 days' worth of dog food.

Applying Fraction Understandings to Nonroutine Problems

Applying Fraction Understandings to Nonroutine Problems

In the following nonroutine problems, you will need to determine which computation(s) to do and how to interpret the computation(s).

Investigation 4.3d – *When Did He Run Out of Gas?*

Jeremy started on a trip from Mobile, Alabama, to New Orleans, approximately 120 miles away. His car ran out of gas after he had gone one-third of the second half of the trip. How far is Jeremy from New Orleans?

Investigation 4.3d – *Discussion*

This is a case in which a good diagram practically solves the problem by itself (see Figure 4.24).

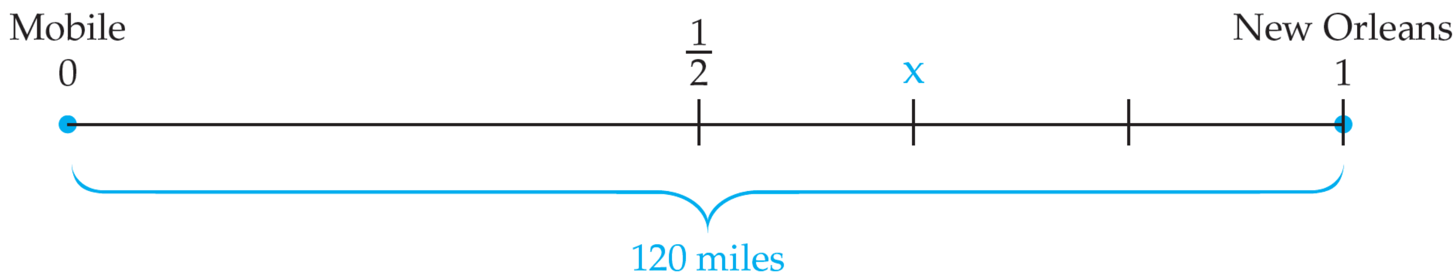


Figure 4.24

If half of the trip is equal to 60 miles, then each of the "thirds of the second half" is equal to 20 miles. X marks the spot of "one-third of the second half of the trip." So Jeremy is still 40 miles from New Orleans.

Investigation 4.3e – *How Many Pieces of Wire?*

A jewelry artisan is making earring hoops. Each hoop requires a piece of wire that is $3\frac{3}{4}$ inches long. If the wire comes in 50-inch coils, how many $3\frac{3}{4}$- Inch pieces can be made from one coil, and how much wire is wasted?

But before you do, ask yourself, “Do I understand the problem? What is the important information in this situation, and how can I apply what I know to solve it? Does the problem’s wording help me devise a plan for solving it? Once I have a solution, can I check it?”

Investigation 4.3e – *Discussion*

Strategy 1: Divide

Some people quickly realize that you can divide “to get the answer.” If you use a calculator, it shows 13.333333. . . . If you use fractions, you get $13\frac{1}{3}$. Many people interpret these numbers to mean that you can get 13 pieces and you will have $\frac{1}{3}$ inch wasted.

Unfortunately, that is not correct. Can you explain why $\frac{1}{3}$ inch is not the correct answer and what the correct answer is?

Investigation 4.3e – *Discussion* continued

One of the reasons why math teachers stress the importance of labels is that they illustrate the meaning of what we are doing. The *meaning* of $13\frac{1}{3}$ is 13 whole hoops and $\frac{1}{3}$ of a hoop. That is, what we have left would make $\frac{1}{3}$ of a hoop. Because one whole piece is $3\frac{3}{4}$ inches long, $\frac{1}{3}$ of a piece is $\frac{1}{3}$ of $3\frac{3}{4}$; that is, $1\frac{1}{4}$ inches is wasted.

How would we check this answer?

One way to check would be to multiply $3\frac{3}{4} \times 13$.

Investigation 4.3e – *Discussion* continued

This would tell us the length of the 13 whole pieces. If this number plus $1\frac{1}{4}$ equals 50, then our answers are correct. In fact, $3\frac{3}{4} \times 13 = 48\frac{3}{4}$ and $48\frac{3}{4} + 1\frac{1}{4} = 50$.

Strategy 2: "Act it out"

Some people understand the problem better when they model the problem with a diagram like that in Figure 4.25.

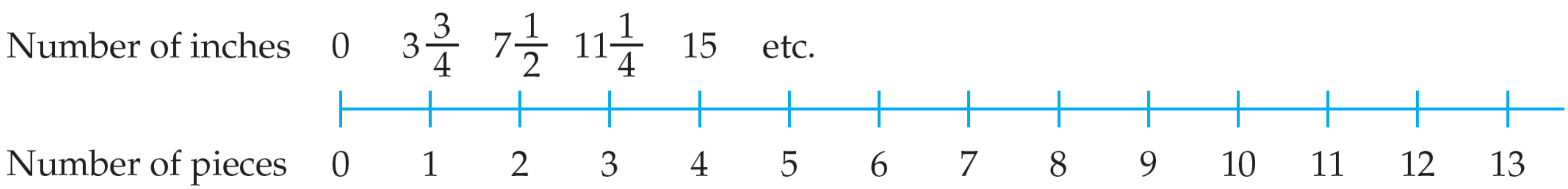


Figure 4.25

Investigation 4.3e – *Discussion* continued

Strategy 3: Make a table

Yet other people solve the problem by starting with one piece and building up as shown in Table 4.6.

Number of pieces	Number of inches	Thinking process
1	$3\frac{3}{4}$	
2	$7\frac{1}{2}$	
3	$11\frac{1}{4}$	
4	15	Using the concept of ratio and proportion, we can reason that if 4 pieces make 15 inches, then 12 pieces would make 45 inches.
12	45	So 1 more piece works.
13	$48\frac{3}{4}$	There are only $1\frac{1}{4}$ inches left, so that is the wasted part.

Table 4.6

Investigation 4.3f – *They've Lost Their Faculty!*

American State College has had to reduce its faculty by $\frac{1}{6}$ because of an economic crisis in the state. The college now has 350 faculty members. A curious reader might ask, "How many did they have before the cut?"

Investigation 4.3f – *Discussion*

Strategy 1: Represent the situation with a diagram

The large rectangle in Figure 4.26 represents the original faculty.

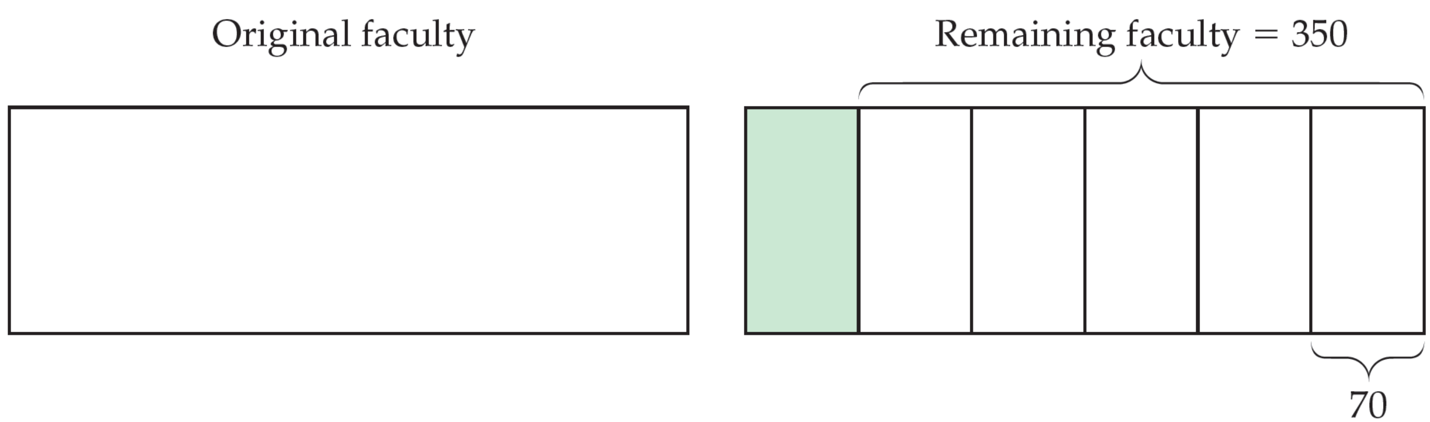


Figure 4.26

Investigation 4.3f – *Discussion* continued

If the college lost $\frac{1}{6}$ of its faculty, then we can divide that rectangle into 6 equal pieces and take away 1 piece, which represents $\frac{1}{6}$ of the faculty. Thus, the 5 remaining boxes represent the remaining $\frac{5}{6}$ of the faculty.

Because we know that there are 350 faculty members now, the value of those 5 boxes is 350. But if the value of 5 boxes is 350, then the value of 1 box is 70. Now we can answer the problem. Do you see why?

Because the value of the original faculty was represented by 6 boxes, there were 6 × 70 = 420 faculty.

Investigation 4.3f – *Discussion* continued

Strategy 2: Use algebra

Can you find an algebraic solution?

One algebraic solution comes from letting *x* = the number of the faculty before the cut. The remaining faculty represents $\frac{5}{6}$ of the original, so 350 is $\frac{5}{6}$ of x; that is, $\frac{5}{6}x = 350$.

We solve the equation by multiplying both sides by $\frac{6}{5}$:

$$\left(\frac{6}{5}\right)\frac{5}{6}x = \left(\frac{6}{5}\right)350 \quad \text{and} \quad x = 420$$