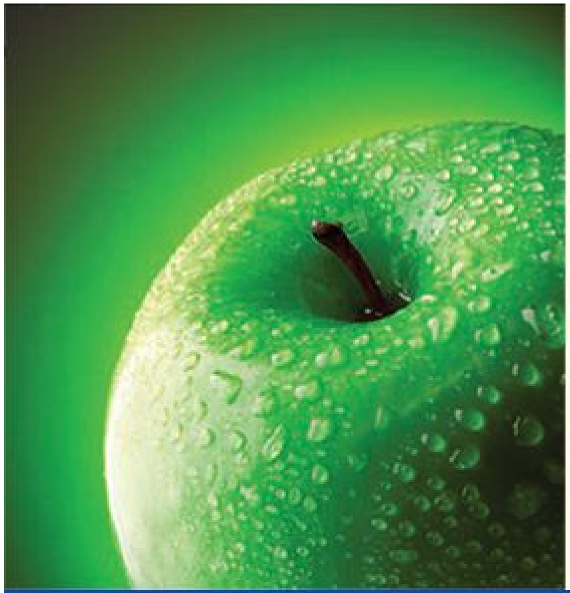

4

Understanding Multiplication and Division

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SECTION

4.2

Understanding Division of Whole Numbers

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What do you think?

- Do you think our multiplication and division algorithms would work with the Roman or Egyptian numeration system?
- When we do long division, we “bring down” the next number. What does *bring down* mean?

Contexts for Division

Contexts for Division

Assume that you are a young child who has not yet learned division. How might you solve these problems, using manipulatives or diagrams, or using common sense and other mathematical knowledge?

1. Applewood Elementary School has just bought 24 Apple computers for its 4 fifth-grade classes. How many computers will each classroom get?
2. Carlos has 24 apples with which to make apple pies. If it takes 4 apples per pie, how many pies can he bake?

Contexts for Division

3. Melissa, Vanessa, Corissa, and Valerie bought a bolt of cloth that is 24 yards long. If they share it equally, how many yards of cloth does each person get?

4. Jeannie is making popsicles. If she has 24 ounces of juice and each popsicle takes 4 ounces, how many popsicles can she make?

Now go back to these four problems and answer the following questions:

- In what ways are the problems different? In what ways are they similar?

Contexts for Division

- What picture or model might a child use to solve each of these? How are the pictures/models and strategies similar and different for each?
- What does division mean? For example, what words come to mind when you think of division?

Some of these problems involve discrete objects (Problems 1 and 2), whereas other problems involve measured (continuous) amounts (Problems 3 and 4).

Contexts for Division

Another difference emerges when we solve these problems as young children do. Left to themselves, most children will solve Problems 1 and 3 in a way that is very different from the way they will solve Problems 2 and 4.

Models for Division

Models for Division

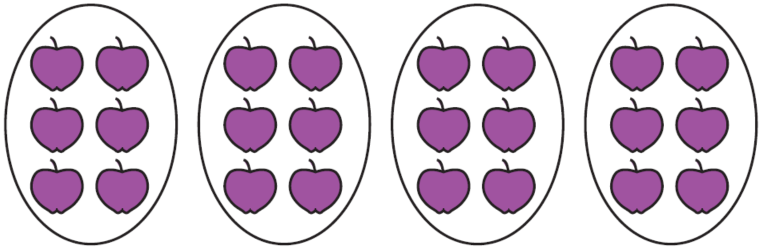
Let us solve Problems 1 and 2 to understand two different models of division.

1. Applewood Elementary School has just bought 24 Apple Computers for its 4 fifth-grade classes. How many computers will each classroom get?

To solve the first problem, many children will give one computer to each class, dealing them out one by one: one for our class, one for your class, and so on.

Models for Division

Visually, we have



Each classroom gets 6 Apple computers.

This model is called the **partitioning model** because we solve the problem by first setting up the appropriate number of groups (partitions), which we then fill, as in one for you, one for you, and so on until they are gone, just like a child might do if sharing a bag of candies.

Models for Division

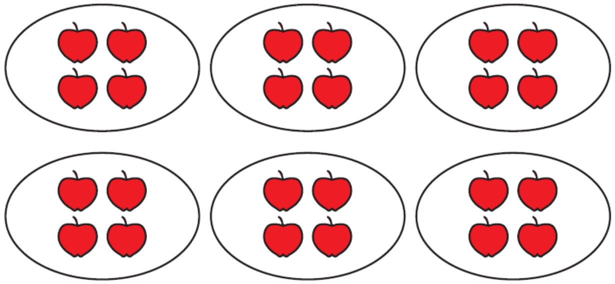
Because of this, sometimes this is called the sharing model.

2. Carlos has 24 apples with which to make apple pies. If it takes 4 apples per pie, how many pies can he bake?

To solve the second problem, most children do something very different. They make groups of 4 (each representing one pie) until there are no more apples left.

Models for Division

Visually, we have



We have enough apples to make 6 pies.

This model is called the **repeated subtraction model** because we repeatedly subtract the given amount until we can do so no more. Some books call this the measurement model of division.

Models for Division

Now let us look at how the models are related. First we need some terminology.

Division terminology

In both cases, the problem would be written as

$$24 \div 4 = 6 \text{ or as } 4\overset{6}{\overline{)24}} \text{ or as } \frac{24}{4} = 6$$

Models for Division

The division of the number *b* by the nonzero number *n* is formally defined as

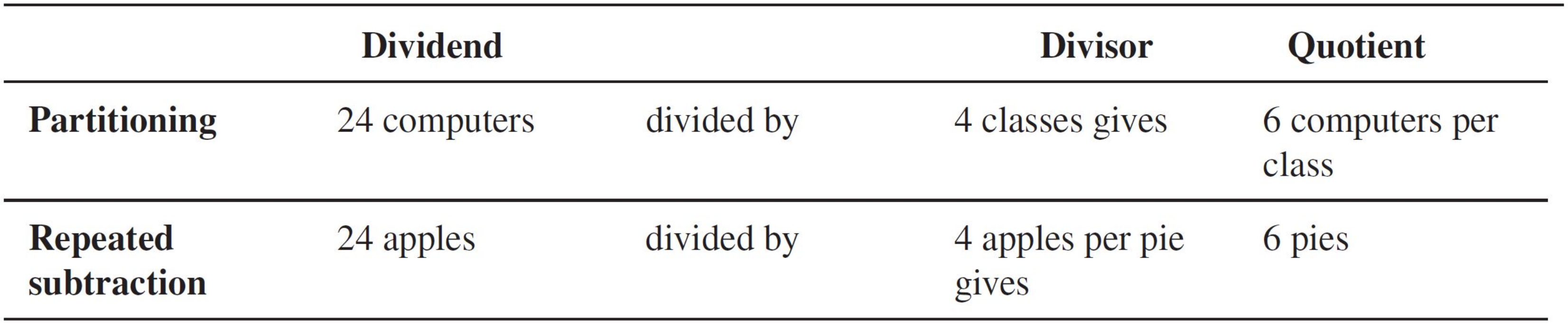
$$b \div n = a \text{ if } a \cdot n = b$$

The number *b* is called the **dividend**, *n* is called the **divisor**, and *a* is called the **quotient**.

Models for Division

Comparing the two division models

With this terminology, we can compare the two models in Table 4.3. What patterns or connections between the two models do you notice?



	Dividend		Divisor	Quotient
Partitioning	24 computers	divided by	4 classes gives	6 computers per class
Repeated subtraction	24 apples	divided by	4 apples per pie gives	6 pies

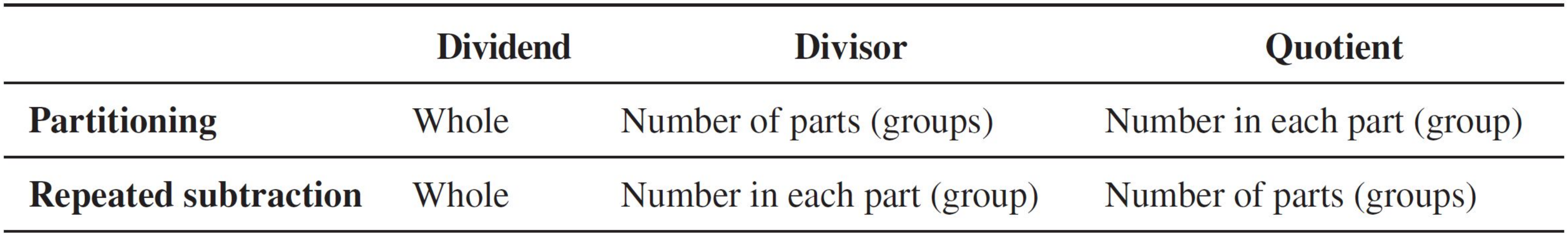
Table 4.3

Models for Division

We can also represent both division contexts with the general part–whole box model we have developed for the other three operations

Models for Division

At the most general level, we have:



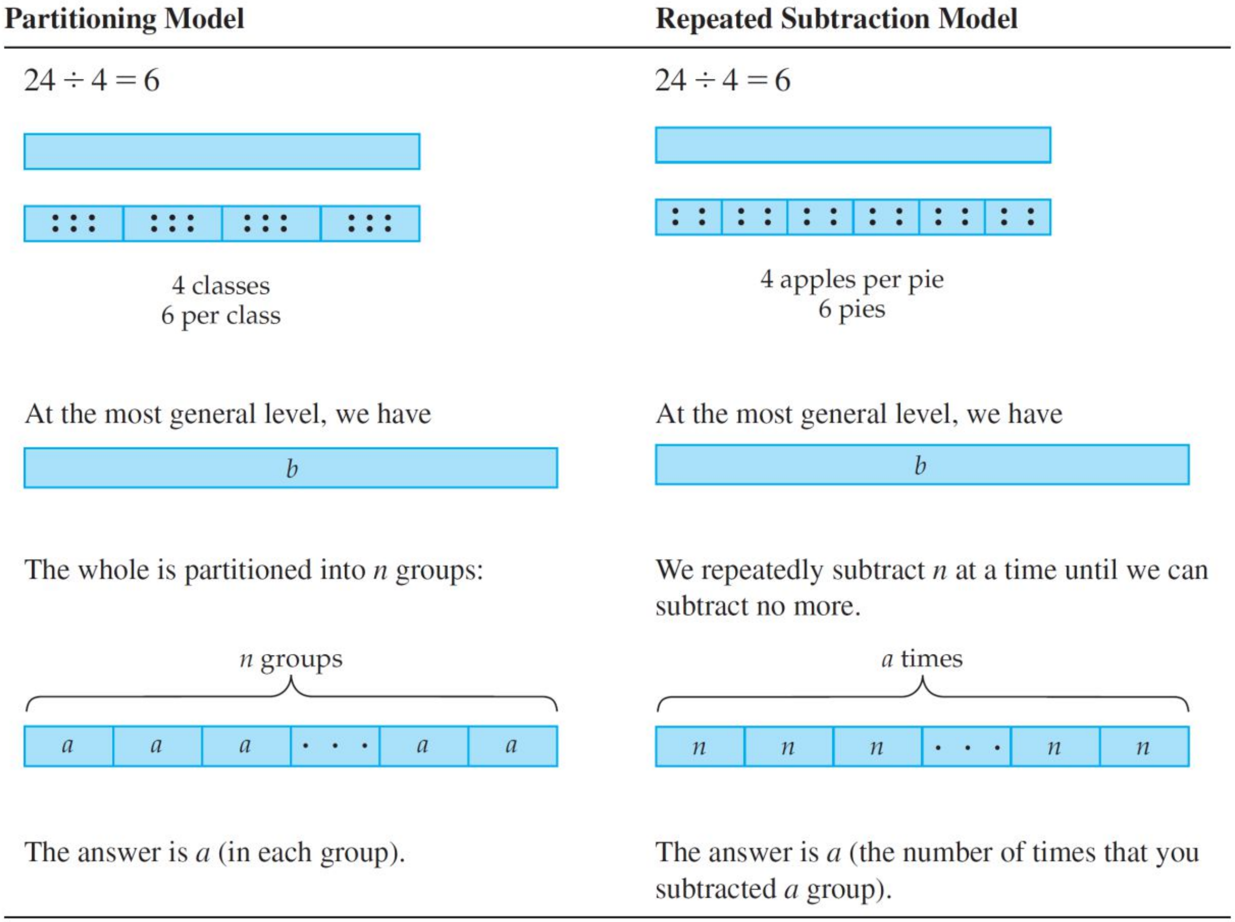
	Dividend	Divisor	Quotient
Partitioning	Whole	Number of parts (groups)	Number in each part (group)
Repeated subtraction	Whole	Number in each part (group)	Number of parts (groups)

Table 4.3 (continued)

Stop for a moment to reflect on these models. How are these two models of division alike, and how are they different?

Models for Division

We can represent the models visually:



Partitioning Model	Repeated Subtraction Model
$24 \div 4 = 6$	$24 \div 4 = 6$
4 classes 6 per class	4 apples per pie 6 pies
At the most general level, we have	At the most general level, we have
The whole is partitioned into n groups:	We repeatedly subtract n at a time until we can subtract no more.
n groups	a times
The answer is a (in each group).	The answer is a (the number of times that you subtracted a group).

The Missing-Factor Model of Division

The Missing-Factor Model of Division

Just as some children interpret and solve some subtraction problems by finding the missing addend, some children interpret and solve some division problems by finding the missing factor.

This model for division, called the **missing-factor model** for division, gives us another tool for working with division.

Representing Division with Number Lines

Representing Division with Number Lines

What problem is represented in the number line in Figure 4.13? Does the number line fit both models or just one model?

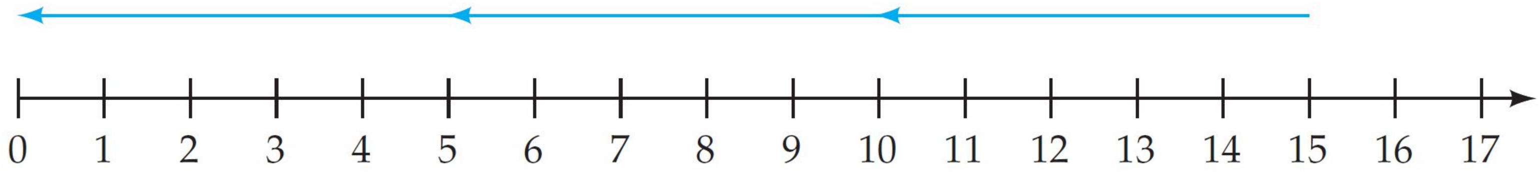


Figure 4.13

If the problem is 15 ÷ 5 = 3, then the number line fits the repeated subtraction model. If the problem is 15 ÷ 3 = 5, then the number line fits the partition model.

Properties of Division

Properties of Division

Of the properties we have investigated for the other operations, identity, commutativity, associativity, and closure, which do you think hold for division?

If you have been making connections among the operations, you probably have realized that addition and multiplication are alike in that each operation has an identity and that the commutative and associative properties hold for those operations.

Properties of Division

You saw that those properties did not hold for subtraction, although subtraction does have a right-identity, such that $a - 0 = a$ but $0 - a \neq a$. Therefore, you might be inclined to say the same for division.

Division does have a right-identity; that is, $a \div 1 = a$ for all numbers. The commutative and associative properties do not hold for division since $3 \div 4 \neq 4 \div 3$ and $(3 \div 4) \div 5 \neq 3 \div (4 \div 5)$.

Properties of Division

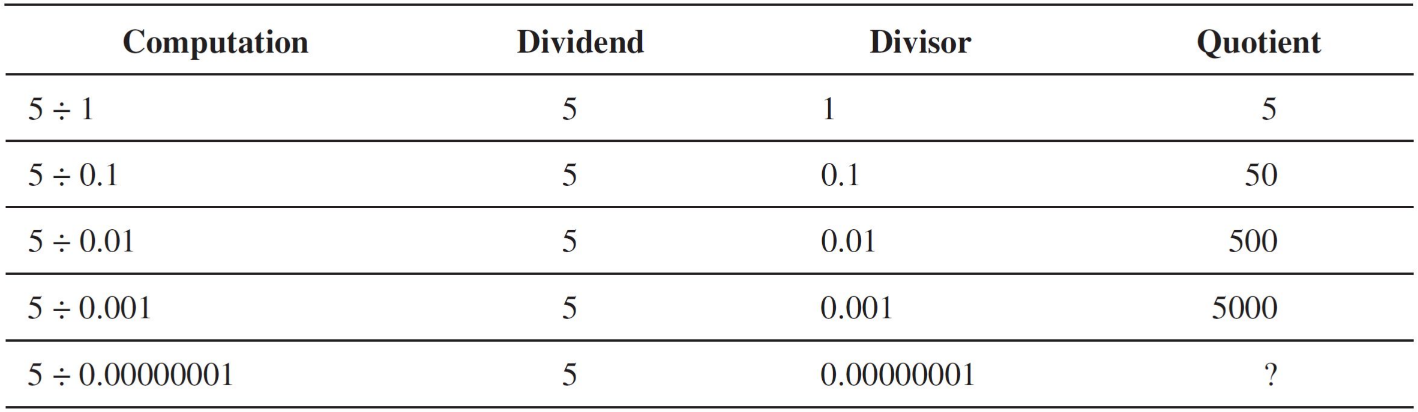
Similarly, the set of whole numbers is not closed under division since sometimes when we divide a whole number by a whole number, such as 3 ÷ 4, it does not equal another whole number.

Division By Zero

Division By Zero

Our definition for division implies that “you can’t divide by zero.” Can you explain why?

There are several ways in which we can investigate this problem. We can use inductive reasoning: Make a table and look for patterns. What pattern do you see in Table 4.4 that can help you to explain why “you can’t divide by zero”?



Computation	Dividend	Divisor	Quotient
5 ÷ 1	5	1	5
5 ÷ 0.1	5	0.1	50
5 ÷ 0.01	5	0.01	500
5 ÷ 0.001	5	0.001	5000
5 ÷ 0.00000001	5	0.00000001	?

Table 4.4

Division By Zero

As the divisor becomes smaller and smaller, the quotient becomes larger and larger, heading to infinity, which we cannot define as a number.

A third approach to this problem is to connect multiplication with division and assume that it is possible; this is called **indirect proof**. If division by zero were possible, then a nonzero x divided by zero would be equal to some number k (that is, $x \div 0 = k$).

Division By Zero

However, if this is true, then according to our definition of division, $k \cdot 0 = x$, but this is not possible because we know that $k \cdot 0 = 0$.

Many students find this proof to be too abstract. We can use an actual example and begin with 0 divided by a number.

$\frac{0}{4} = 0$ because $4 \cdot 0 = 0$

Division By Zero

If it were possible to divide by 0, then $\frac{4}{0}$ would equal some number; we will call this number *x*.

$$\frac{4}{0} = x$$

Division By Zero

Because the relationship with multiplication must stay true, this means that $0 \cdot x = 4$. This is impossible because 0 times any number is 0.

Investigation 4.2a – *Mental Division*

Examine each of the problems below carefully. How might you determine the exact answer, applying what you know about division? Briefly note the strategies you used, and try to give names to them.

1. $20\overline{)6000}$
2. $400\overline{)20,000}$
3. $8\overline{)152}$
4. $5\overline{)345}$

Investigation 4.2a – *Discussion*

In Problem 1, we will examine two different ways to use **canceling**.

One way to cancel in Problem 1 is to transform it from $\frac{6000}{20}$ to $\frac{600}{2}$, which is equal to 300.

$$\frac{6000}{20}=\frac{600\cdot 10}{2\cdot 10}=\frac{600\cdot \overset{1}{\cancel{10}}}{2\cdot \underset{1}{\cancel{10}}}=\frac{600}{2}=300$$

Investigation 4.2a – *Discussion* continued

In this case, we are applying the idea of **equivalent fractions**, which we explored in Section 2.2 on fractions. That is, we are dividing both numerator and denominator by the same amount, 10.

Another method of canceling transforms the problem from $\frac{6000}{20}$ to $\frac{(60 \cdot 100)}{20}$, which can then be simplified to 3 · 100 because 20 divides both 60 and 100 without remainder.

$$\frac{6000}{20} = \frac{60 \cdot 100}{20} = \frac{\overset{3}{\cancel{60}} \cdot 100}{\underset{1}{\cancel{20}}} = 300$$

Investigation 4.2a – *Discussion*

continued

You can use each of the above strategies in Problem 2. Try them for yourself. Problem 3 lends itself to compatible numbers; for example, 160 ÷ 8 = 20. We can use the distributive property to get an exact answer, since

$$\frac{152}{8} = \frac{160-8}{8} = \frac{160}{8} - \frac{8}{8} = 20 - 1 = 19$$

We can also use the distributive property in Problem 4:

$$\frac{345}{5} = \frac{350-5}{5} = \frac{350}{5} - \frac{5}{5} = 70 - 1 = 69$$

Investigation 4.2a – *Discussion* continued

In Problem 4, we can also use the idea of equivalent fractions and multiply both numerator and denominator by 2. The resulting fraction, $\frac{690}{10}$, easily simplifies to 69.

Division Algorithms

Investigation 4.2b – *Understanding Division Algorithms*

Solve the two division problems as though you didn't know any algorithms. You may use base ten blocks or draw diagrams or use reasoning.

A. Warren has 252 guests coming to his wedding. Each table holds 4 guests. How many tables will he need?

B. Mickey has 252 marbles that he wants to distribute equally into 4 piles. How many marbles are in each pile?

Investigation 4.2b – *Discussion*

A. This is a repeated subtraction problem, meaning that we need to put the 252 into groups of 4. This can be done with base ten blocks, as we will show in question B, except here we would put the blocks into groups of 4.

However, in this context, the model is not as intuitive. Some students solve this problem by using the missing factor model: 4 times what makes 252?

Guess	**Result**	**Thinking**
50 tables	200 people	Too low, try a bigger number, let's say 60 tables.
60 tables	240 people	We need 3 more tables for the 12 remaining people.
63 tables	252 people	

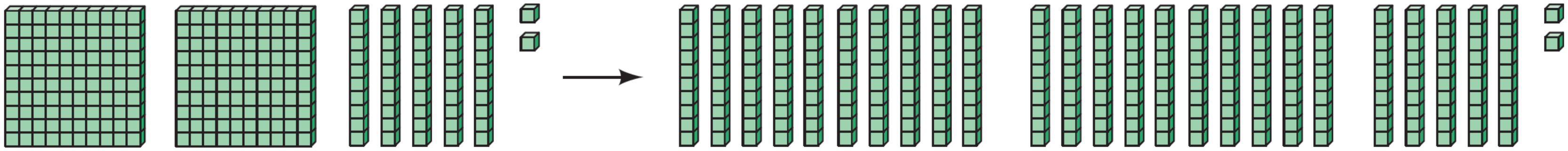
Investigation 4.2b – *Discussion* continued

B. The base ten blocks are more useful for Problem B. In this case, Mickey's problem is to divide (distribute) 252 into four groups equally. Using base ten manipulatives, he can figure out how to do this most efficiently.

Because he does not have enough hundreds to give a flat to each person, he has to trade the 2 hundreds (flats) for 20 longs. He still has 252, but physically he has 25 longs and 2 singles.

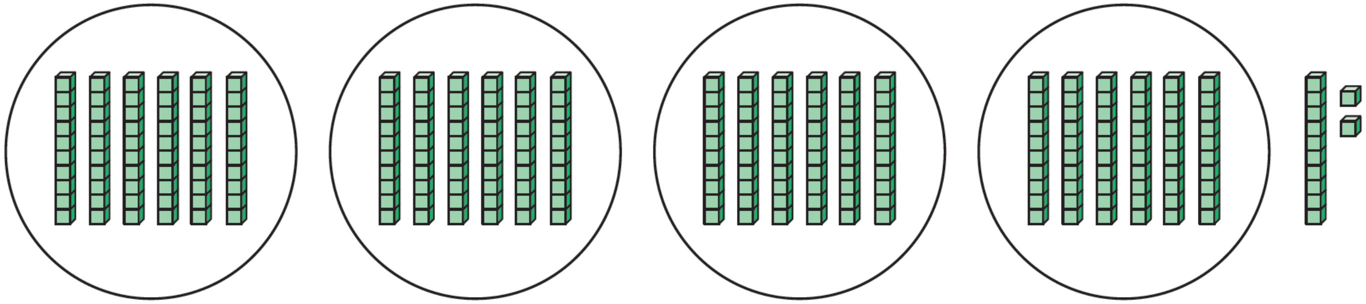
Investigation 4.2b – *Discussion*

continued



Now he can place 6 longs in each group. The corresponding abstract step is shown at the left.

$$\begin{array}{r} 6\\ 4\overline{)252}\\ \underline{24}\\ 1\end{array}$$



Investigation 4.2b – *Discussion* continued

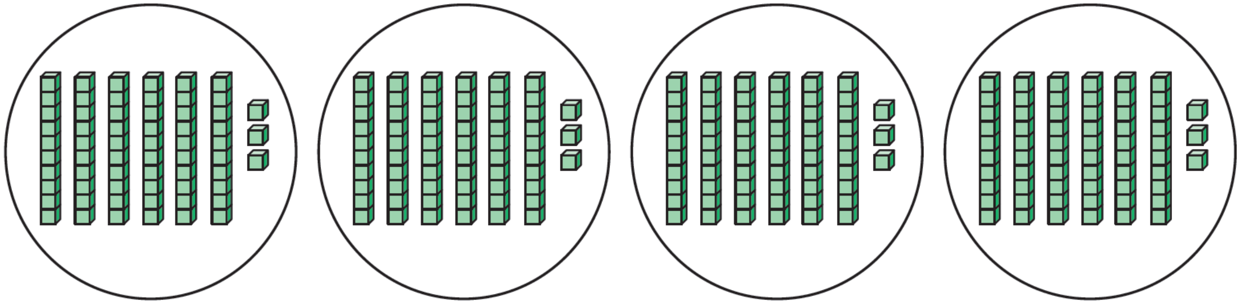
Take a moment to compare the abstract algorithm on the left to the pictorial representation with the blocks. It is important to connect them. Where is the 6 that is above the division bar? Where is the 24 in the picture, and the “1” we get when we subtract?

The 6 represents the 6 longs that we can put in each of the 4 groups. The 24 represents the 24 longs that are total in these 4 groups of 6, leaving 1 long left over.

Investigation 4.2b – *Discussion* continued

He is left with 1 long and 2 singles, representing 12; thus, he trades the long for 10 singles. This lets him place 3 units in each group. The corresponding symbolic step in shown at the left.

$$\begin{array}{r} 63 \\ 4\overline{)252} \\ \underline{24} \\ 12 \\ \underline{12} \\ 0 \end{array}$$



Investigation 4.2c – *The Scaffolding Algorithm*

Another algorithm, which has enjoyed favor in different parts of the world over the centuries and is gaining in popularity in the United States, is called the **scaffolding algorithm**. Two representations of this algorithm are shown for 4356 ÷ 6 in Figure 4.14.

$$\begin{array}{r} 6\\ 20\\ 200\\ 500\\ 6\overline{)4356}\\ \underline{3000}\\ 1356\\ \underline{1200}\\ 156\\ \underline{120}\\ 36\\ \underline{36} \end{array}$$

$$\begin{array}{rl} 6\overline{)4356} & 500\\ \underline{3000} & \\ 1356 & 200\\ \underline{1200} & \\ 156 & 20\\ \underline{120} & \\ 36 & 6\\ \underline{36} & \end{array}$$

Figure 4.14

Investigation 4.2c – *Discussion*

Let us examine how it works. Consider the problem $6\overline{)4356}$. A child with poor multiplication facts will often have trouble with the first step in the traditional algorithm. If the child chooses wrong, such as 6 × 6 = 36, then the whole problem is doomed (Figure 4.15).

$$\begin{array}{r}
699\text{ R }162 \\
6\overline{)4356} \\
\underline{36} \\
75 \\
\underline{54} \\
216 \\
\underline{54} \\
162
\end{array}$$

Figure 4.15

Investigation 4.2c – *Discussion* continued

The alternative approach is to look at the problem from a different vantage point. Instead of asking how many 6s are in 43 or 6 × what is close to 43, we ask the child to look at the problem in context.

For example, if a company produces 4356 bottles of soda each day, how many six-packs would this be? Then ask the child to estimate. Let’s say the estimate is 500. The child determines how much 500 six-packs would be: 500 × 6 = 3000.

Investigation 4.2c – *Discussion* continued

This guess is recorded above the problem (see the work shown above), and then what remains after this guess is determined: Subtracting from 4356, we have 1356 left. Now the question is: How many six-packs would this be?

Using the knowledge from the previous guess, 200 is a good guess, and 200 × 6 = 1200. Subtracting again, this gives us 156 left. Let’s say the child guesses 20; this uses up 120 and leaves 36, at which point the child might guess 6, and there would be 0 left.

Investigation 4.2c – *Discussion* continued

The final answer is found by adding the numbers we got when we divided, 500 + 200 + 20 + 6 = 726. Note that this example shows only one of many possible solution paths for 4356 ÷ 6 using the scaffolding algorithm.

Investigation 4.2d – *Children's Mistakes in Division*

The error below is one that so many of my students have seen and is one reason that many elementary school textbooks present the scaffolding algorithm. What misunderstanding(s) of place value and/or division might be causing the problems?

$$\begin{array}{rl} 699 & \text{R } 162 \\ 6\overline{)4356} & \\ \underline{36} & \\ 75 & \\ \underline{54} & \\ 216 & \\ \underline{54} & \\ 162 & \end{array}$$

Investigation 4.2d – *Discussion*

In this case, there are likely two problems: weak division facts and not remembering that in each step the remainder must be less than the divisor. When the student had a remainder of 7 in the first step, that led to the remainder that was too big.

How could the scaffolding method be used at this point to find the answer? We can take the remainder of 162 and divide it by 6, to get 27. Adding this to the 699 would give the final answer of 726.

Investigation 4.2e – *Estimates with Division*

A. Mr. and Mrs. Smith pay $9100 a year for their son's college. Translate this into a monthly payment that they can put into their budget.

Investigation 4.2e – *Discussion*

Using the **missing-factor model**, ask yourself, “12 times what is closest to 91?” 12 × 7 = 84, 12 × 8 = 96. Because 91 is just about in the middle (of 84 and 96), it is reasonable to conclude that $12 \times 7\frac{1}{2}$ will be close to 91, and so our estimate is $7\frac{1}{2}$ hundred a month, or, more conventionally, $750 per month.

Investigation 4.2e – *Estimates with Division*

B. Pierre just bought a new van, and he wants to see what kind of mileage he gets. He filled up with gas after going 489 miles, and the car took 19 gallons of gas. Estimate the mileage.

Investigation 4.2e – *Discussion*

One strategy is to round both the divisor and the dividend up.

$\frac{489}{19}$ will be close to $\frac{500}{20}$, and this is equivalent to $\frac{50}{2}$, and so he is getting about 25 mpg.

This strategy is interesting because it is not identical to the one used in multiplication, where we get more accurate estimates if we round one number up and the other down.

Investigation 4.2e – *Discussion*

continued

In the case of division, we obtain more accurate estimates if we round both numbers up or both numbers down to manageable numbers. Can you explain why?

Hint: It might be helpful to make up some problems and check this out. For example, try 3750 ÷ 13. Decrease both: to 3600 and 12, which are compatible numbers. Increase both: to 3900 and 13. Then decrease one and increase the other—for example, 3750 to 4000 and 13 to 10.

Investigation 4.2e – *Discussion* continued

Compare the results. Make up some other problems. Can you use ideas developed thus far to help you?

Explanation: We can see that rounding 489 up increases the quotient, that is, $\frac{489}{19} > \frac{500}{19}$.

We can see that rounding 19 up decreases the quotient, that is, $\frac{489}{19} > \frac{489}{20}$.

Investigation 4.2e – *Discussion* continued

Thus, rounding both numbers up has the effect of canceling their effects, similar to the canceling effect in multiplication when we round one number up and the other down in 59 × 41.

Rounding both of the numbers down produces the same canceling effect.

Investigation 4.2e – *Estimates with Division*

C. A large business ran 5432 copies in the last 7 business days. How many copies is it averaging per day?

Investigation 4.2e – *Discussion*

A rough estimate could be obtained by making compatible numbers, increasing the 5432, and solving the problem. For example, 5600 ÷ 7 gives an estimate of 800.

Another strategy would be to use multiplication:

$$7 \times 700 = 4900$$

$$7 \times 800 = 5600$$

Because 5432 is closer to 5600 than it is to 4900, a rough estimate would be just under 800.

Investigation 4.2e – *Estimates with Division*

D. Employees put 25¢ in a can for each cup of coffee they drink. Last week the can contained $29.50. How many cups of coffee were drunk?

Investigation 4.2e – *Discussion*

Strategy 1: Solve the problem in dollars

That is, 25¢ per cup means 4 cups per dollar.

You can estimate $30 × 4 cups per dollar = 120 cups.

Strategy 2: Solve the problem in cents

That is, $29.50 is about $30, which is equivalent to 3000¢.

$$\frac{3000}{25}=\frac{2500+500}{25}=\frac{2500}{25}+\frac{500}{25}=100+20=120$$

Investigation 4.2f – *Number Sense with Division*

A. What is the value of the divisor that makes this division problem true?

$$\begin{array}{r} 11 \text{ R } 1 \\ \overline{)463} \end{array}$$

Investigation 4.2f – *Discussion*

Think 11 × ? is less than 463? Because 11 × 40 = 440, we can try 11 × 41, which is 451, and then 11 × 42, which is 462, which gives a remainder of 1.

Investigation 4.2f – *Number Sense with Division*

B. Take no more than 5 to 10 seconds to determine whether this answer is reasonable. That is, make your determination using number sense, without doing any pencil-and-paper work.

$$\begin{array}{r} 704 \\ 62\overline{)4268} \end{array}$$

Investigation 4.2f – *Discussion*

Most people find it easier to translate this to a multiplication problem: 704 × 62. We quickly see that 700 × 60 = 42,000. Thus, the answer is not reasonable.

Investigation 4.2f – *Number Sense with Division*

C. Determine the three missing digits:

$4 \square \square 2 \div 8 \square = 48$

Investigation 4.2f – *Discussion*

One of the key themes in this book is the idea of multiple representations. Sometimes, one representation sheds more light on a problem than another. If you were stuck, look at the following representation of the problem. Does this help?

$$8\,\square \times 48 = 4\,\square\,\square\,2$$

What does this tell you now about the ones digit of the first number?

Investigation 4.2f – *Discussion* continued

It must be a 4 or a 9. Do you see why? Test your intuition—which one do you think it is?

We have to get a 2 in the ones place of the answer when we multiply the 8 in the ones place of the 48 times the missing ones place we are looking for. This only happens when we do 8 × 4, or 8 × 9.

Operation Sense

Operation Sense

Now that we have examined each of the four operations, let us explore some problems in which we can apply our knowledge. Someone who knows which operation to perform in a problem is said to have good **operation sense**.

Operation sense is more than just knowing what operation to use in what situation. Students with good operation sense have the following “knowledge”:

- They can see the relationships among the operations—for example, multiplication is the inverse of division.

Operation Sense

- They can apply their understanding of the properties of the operations—for example, they can use the distributive property.
- The diagrams that they draw connect the problem to the models of the operations.
- They have a sense of the effects of each operation—for example, the larger the divisor, the smaller the quotient.

Divisibility

Divisibility

Divisibility is an interesting extension of operation and number sense, so let’s look at this concept now. This will help deepen our understanding of place value and of division. We can say that 3 divides 12 because 12 is evenly divisible by 3 (that is, there is no remainder).

We can describe this relationship in other ways as well:

3 is a factor of 12
3 divides 12

Divisibility

12 is divisible by 3

12 is a multiple of 3

Symbolically, we write this relationship as $3|12,$ which is read as “3 divides 12.”

Investigation 4.2g – *Interesting Dates*

The date March 4, 2012 can be written as 3/4/12, which is mathematically interesting because 3 times 4512. Another such date is June 2, 2012 (6/2/12).

Determine all the dates with this relationship in the year 2020, and in the year 2023.

Investigation 4.2g – *Discussion*

In the year 2020, there were five dates with this relationship:

1/20/20, 2/10/20, 4/5/20, 5/4/20, 10/2/20

What about the year 2023? There was only one date with this relationship, 1/23/17.

What is different about the number 20 and the number 23 that the year 2020 has more than the year 2023? Why does the year 2023 only have one such date? The number 23 is a prime number since its only factors are 1 and 23.

Investigation 4.2g – *Discussion*

continued

The number 20 is a composite number because it is composed of more than 2 factors.

We define a prime number as a number with exactly 2 factors and a composite number as a number with more than 2 factors. Working with these definitions, is the number 1 a prime or a composite?

The number 1 only has 1 factor (1), so it does not meet either the prime number or the composite number definition, so it is neither.

Recognizing Divisibility

Recognizing Divisibility

Exploring divisibility will help us to develop number and operation sense, as well as to develop our understanding of place value. This is also a useful tool when developing examples to use in your classroom. We will explore patterns and visuals that will help us understand divisibility by 5, 10, 2, 4, 8, 3, and 9.

Divisibility by 5

How can we tell by looking at a number whether it is divisible by 5?

Recognizing Divisibility

Patterns

Recognizing whether a number is divisible by 5 is something that many elementary students observe when skip counting by 5.

Let’s consider numbers that are divisible by 5 and see the pattern:

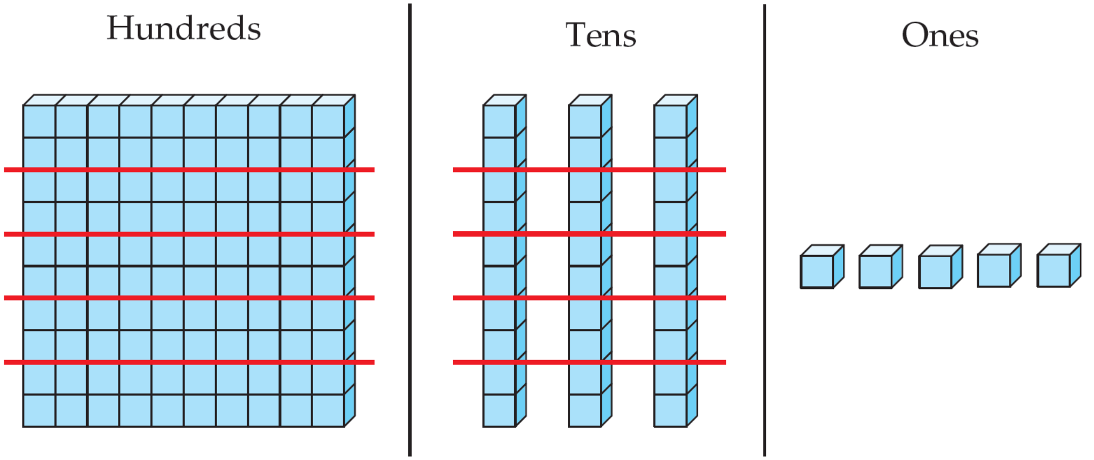
5, 10, 15, 20, 25, 30, …

What do all of these numbers have in common? The ones digit is either a 0 or a 5.

Recognizing Divisibility

Visually

Base ten blocks will help us to see all of the divisibility ideas. With 5, why are we only concerned with the ones digit? Why does it not matter how many tens, hundreds, and so on we have in the number? Let’s use the number 135 to illustrate.



Recognizing Divisibility

Ten is divisible by 5, so it doesn’t matter how many tens there are, we will be able to divide each ten “long” by 5. One hundred is divisible by 5, so it doesn’t matter how many hundreds there are, we will be able to divide them by 5. This same reasoning is true for all the place values to the left. This leaves us with only needing to consider the ones place. If there are 5 or 0 ones there, we will be able to divide by 5.

Therefore, a natural number is divisible by 5 if the ones digit is either a 0 or a 5.

Recognizing Divisibility

Divisibility by 10

Patterns

Just like 5, elementary students will often observe this pattern fairly easily.

The first numbers divisible by 10 are:

10, 20, 30, 40, 50, …

The pattern points to the fact that a number is divisible by 10 if it has a 0 in the ones place.

Recognizing Divisibility

Visually

Similar to 5, the ten long, hundred flat, thousand cube, and so on, are all divisible by 10. So, here again we are only concerned with how many are in the ones place. Only if there is a 0 in the ones place will the number be divisible by 10.

Therefore, a natural number is divisible by 10 if there is a 0 in the ones place.

Recognizing Divisibility

Divisibility by 2

Patterns

What do the numbers that are divisible by 2 have in common?

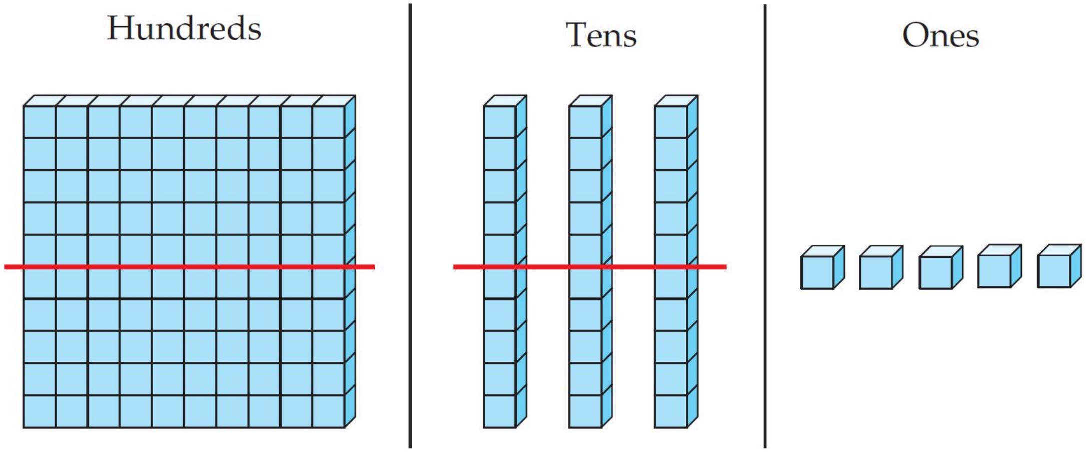
2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, …

The pattern we can see here is that all of the numbers have a 0, 2, 4, 6, or 8 in the ones place. In other words, the ones place is an even number, or the ones place is divisible by 2.

Recognizing Divisibility

Visually

Consider how we looked at the divisibility for 5 and 10 visually and consider why with 2 we also are not concerned with any digit other than that in the ones place. Why is that?



Recognizing Divisibility

Just like with 5 and 10, 10 is divisible by 2, 100 is divisible by 2, and so on, so we only need to look at how many ones we have.

Therefore, a natural number is divisible by 2 if the ones place is divisible by 2.

Recognizing Divisibility

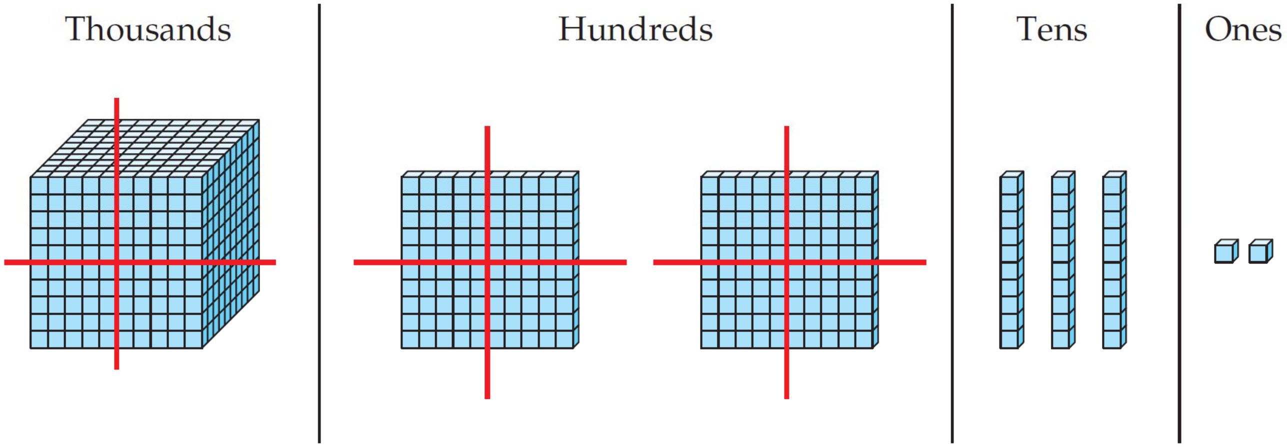
Divisibility by 4

The pattern of this one does not jump out as easily, so let’s consider it visually and connect it to what we did above. Which place values are divisible by 4? Ten is not, but 100 is divisible by 4, and so is 1000, and above.

So, it does not matter how many hundreds, thousands, and so on, that we have, we will be able to divide those by 4. But since 10 is not divisible by 4, we need to look at both the tens and ones places.

Recognizing Divisibility

Let’s use 1232 as an example.



Since 32 is divisible by 4, we know that 1232 is divisible by 4. Therefore, a natural number is divisible by 4 if the number of singles in the tens and ones place is divisible by 4.

Recognizing Divisibility

For example, 1,654,316 is divisible by 4 since 16 is divisible by 4.

Is 3,215 divisible by 4? No, because 15 is not divisible by 4.

Divisibility by 8

How can we apply what was just discussed for 4 to create a divisibility tool for 8? Ten is not divisible by 8, nor is 100. However, 1000 is divisible by 8, and 10,000, and so on.

Recognizing Divisibility

Therefore, a natural number is divisible by 8 if the number of singles in the number represented by the hundreds, tens, and ones place is divisible by 8.

For example, 1,567,168 is divisible by 8 since 168 is divisible by 8.

Is 14,215 divisible by 8? No, because 215 is not divisible by 8.

Recognizing Divisibility

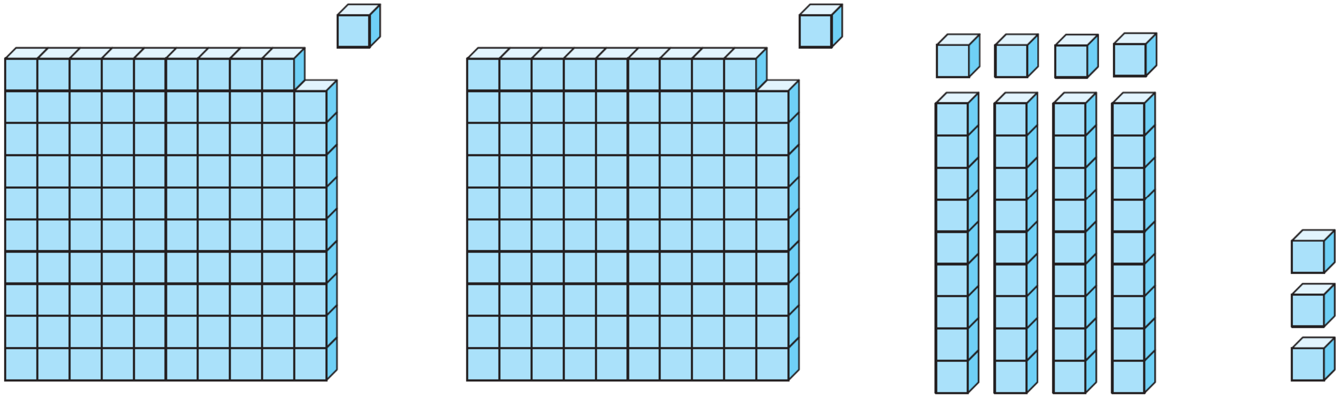
Divisibility by 3

What happens when we divide each place value by 3? When we divide a long of 10 by 3, we will have one left over from each long because 9 is divisible by 3; when we divide a flat of 100 by 3, we will have one left over from each flat because 99 is divisible by 3.

The same pattern holds true for the thousands, ten thousands, and so on.

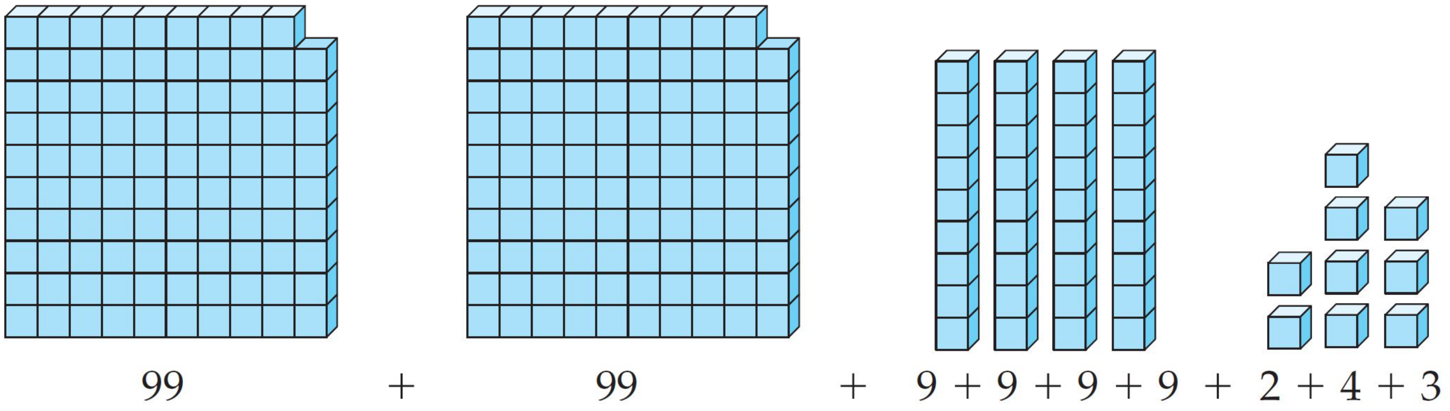
Recognizing Divisibility

Let's look at 243 as an example.



If we take all of these leftovers and put them in a set, we would have 2 left from the hundreds place of 243, 4 left from the tens place, and the 3 ones in the number.

Recognizing Divisibility



Since the 99s and the 9s are divisible by 3, we only need to look at the 2141359 leftovers. Since these 9 leftovers are divisible by 3, then the entire number is also divisible by 3.

Recognizing Divisibility

In each place value, the number of leftovers will be the same as the digit in the place value, since one single is left from each flat, long, and so forth.

Therefore, a natural number is divisible by 3 if the sum of the digits is divisible by 3.

For example, 1,516,113 is divisible by 3 since 18 is the sum of the digits and 18 is divisible by 3. Is 2,347,843 divisible by 3?

No, because when we add the digits, we get 31, which is not divisible by 3.

Recognizing Divisibility

Divisibility by 9

The number 9 works similarly to 3. Why would that be?

Just like with 3, when each place value is divided by 9, there will be 1 left over for each block in that place value.

Therefore, a natural number is divisible by 9 if the sum of the digits is divisible by 9.

For example, 1,516,113 is also divisible by 9 since the sum of the digits is 18, which is divisible by 9.

Recognizing Divisibility

Now you have tools for recognizing whether a number is divisible by 5, 10, 2, 4, 8, 3, and 9, and you have seen why these tools work.

Investigation 4.2h – *Using Counterexamples*

Is the following statement true?

If a number is divisible by 2 and divisible by 6, then it is also divisible by 12.

Investigation 4.2h – *Discussion*

Here we can look at examples and see if we can find a “counterexample,” that is, an example where this statement is not true.

What about 24? Divisible by 2 and by 6 and also 12.

What about 36? Divisible by 2 and by 6 and also 12. Looks good.

What about 30? Divisible by 2 and by 6, but not by 12. So we found a counterexample, and therefore the statement is not true.

Investigation 4.2h – *Discussion* continued

This illustrates how counterexamples can help us to make the argument that this statement is not true.

Investigation 4.2i – *Applying Models to a Real-Life Situation*

A teacher is going to do a project with the two fifth-grade classes in the school. One class has 21 students, and the other has 15 students. She has decided that all of these students will be divided into 12 groups and that each group will need 72 inches of string. She has a roll of string that is 882 inches long. Does she have enough string, or will she need to buy more? Do this problem yourself without using a calculator before reading on. . . .

Investigation 4.2i – *Discussion*

You can use either multiplication or division to solve this problem. Using multiplication, $72 \cdot 12 = 864$ tells you that you will have enough string for each group, with 18 inches of string to spare. If you use division, the problem can be interpreted either as a partitioning problem or as a repeated subtraction problem. Do you see why?

- If you divide 882 inches by 72 inches per group, this says that you are seeing the problem in terms of repeated subtraction. What does your answer of 12 with a remainder of 18 tell you?

Investigation 4.2i – *Discussion* continued

- If you divide 882 inches by 12 groups, this says that you are seeing the problem in terms of partitioning. What does your answer of 73 with a remainder of 6 tell you?

In the repeated subtraction model, the 12 says that if you repeatedly cut 72 inches, you can do this 12 times, and you will have 18 inches of string left over. In the partitioning model, the 73 says that if each group is to have an equal amount of string, you have enough string to make 12 pieces, each of which is 73 inches long, and you will have 6 inches left over.

Investigation 4.2i – *Discussion* continued

In either case, you have to understand what your quotient means, and you have to think about what the remainder means.

Although the mathematical answer is that the teacher has enough string, the teacher might actually conclude from her computations that she needs more string. Why is this?

What if the students make mistakes? Because $72 \cdot 12 = 864$, the teacher has only 18 inches of string in reserve.

Investigation 4.2i – *Discussion*

continued

Thus, although the mathematical answer is that the teacher has enough string, the real-life answer would depend on the nature of the project. For example, if the students were to cut the pieces into specified lengths and I anticipated that some groups would mismeasure the lengths, I would not feel confident that I had enough string.

Investigation 4.2j – *Operation Sense*

A. Select the correct operations to make the sentence true. You can use an operation more than once.

Investigation 4.2j – *Discussion*

A few rounds of guess–check–revise generally convince most students that addition and subtraction are not possible. If you explore multiplication and division, trying the multiplication 6 × 5 reveals that ÷ 3 will produce an answer of 10.

Investigation 4.2j – *Operation Sense*

B. Make a number sentence that will produce the desired answer once the unknown amounts are known.

Germaine bought ***A*** shirts for ***B*** dollars each. He sold all of the shirts for a total of ***C*** dollars. How much profit did he make?

Investigation 4.2j – *Discussion*

If we think about it, we can see the repeated addition in a group of shirts with the same price. In selling, he is taking away what he paid for the shirts ($A \times B$); assuming he is making a profit, the C comes first. That is, the number sentence is $C - (A \times B)$.

Recognizing Divisibility

Order of Operations

Complex computations that use more than one operation pose a potential problem. To see what I mean, do this problem with pencil-and-paper and then on your calculator: $3 \cdot 4 - 8 \div 2$. What did you come up with?

If you do this on most calculators, you will get 8. However, if you do each operation in the order in which it appears in the problem or on very basic calculators or many cell phone calculators, you will get 2. Why did the calculator give a different answer?

Recognizing Divisibility

The fact that we can get two different answers from the same problem has led to rules called the **order of operations**. These rules tell you the order in which to perform operations so that each expression can have only one value. Most calculators are programmed to obey the order of operations, though some cell phones are not.