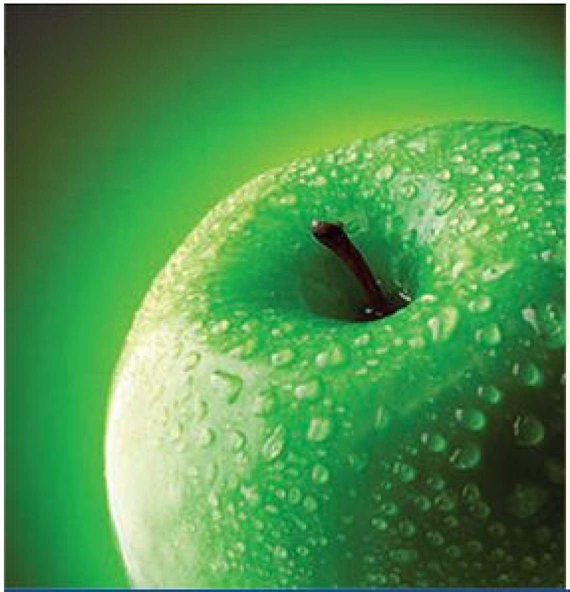


4 Understanding Multiplication and Division

SECTION 4.1

Understanding Multiplication of Whole Numbers

What do you think?

- What other meanings are there for multiplication beyond “repeated addition”?
- Why do we “move over” when we multiply?
- Are there more even or more odd numbers in the multiplication table?

Understanding Multiplication of Whole Numbers

One of the goals of the following examination of multiplication is for you to understand the different meanings of multiplication and why it is useful to know these different meanings.

Contexts for Multiplication

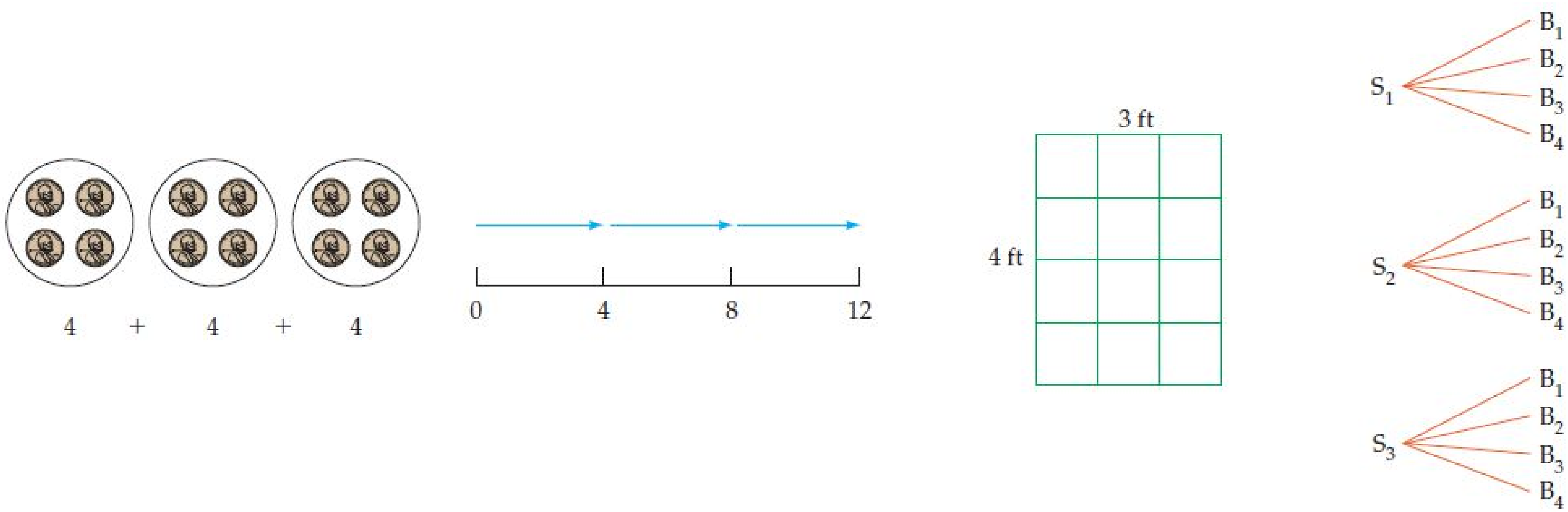
Contexts for Multiplication

First write your own word problem for 4 times 3. Assume that you are a young child who has not yet learned multiplication. How might you solve this problem?

What picture or model might a child use to help them figure out the answers?

Contexts for Multiplication

If we represent the four problems with diagrams, as in Figure 4.1, the differences appear to be tremendous, even though all four problems can be solved as 3 times 4.



Pictorial Representations of 3 × 4 = 12

Figure 4.1

Contexts for Multiplication

As in addition and subtraction, we can use discrete set models and linear measurement models to understand multiplication.

We also have area models and Cartesian product models in multiplication. Consider each of the models below for the solutions to the problems and consider how each shows that $3 \times 4 = 12$.

Contexts for Multiplication

1. One piece of candy costs 4¢. How much would 3 pieces cost?

Problem 1 is like the discrete sets we encountered with addition and subtraction. It is also literally **repeated addition**; it is solved by adding 4 + 4 + 4.

2. If Jackie walks at 4 mph for 3 hours, how far will she have walked?

Problem 2 involves a certain number of measured units and can be represented with a number line that shows repeated addition of the measures.

Contexts for Multiplication

3. A carpet measures 4 feet by 3 feet. What is the area of the carpet?

Problem 3 involves measures, but it involves repeated addition less than it involves counting the number of new units. This problem illustrates the area context of multiplication.

4. Carla has 4 blouses and 3 skirts. How many combinations can she wear?

(Assume that all possible combinations go together!)

Contexts for Multiplication

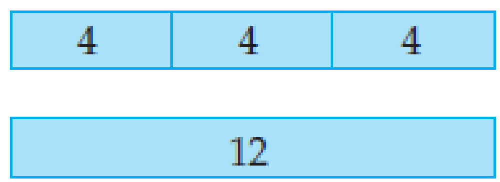
Problem 4 involves discrete objects, but making sense of it relies not on repeated addition but on putting combinations in an array. In set language, what we did in this problem was to examine two sets and look at all possible ways of pairing the elements of those sets.

This visual model is called a tree diagram, which we will see again in our explorations of probability.

A General Model for Multiplication

A General Model for Multiplication

Can you describe the four problems in such a way that they all have something in common, as we did for addition and subtraction? Our general model for addition can be extended to multiplication. Figure 4.2 shows one diagram for 3 times 4.



$3 \times 4 = 12$

Figure 4.2

A General Model for Multiplication

Do you see the resemblance between the addition and multiplication models? As with addition and subtraction, the *general model of multiplication* can be cast in part–whole language: The product (the whole) is built from parts that are equal in size or amount.

Traditionally, multiplication is represented as

$$n \text{ times } a = \overbrace{a + a + a + \cdots + a}^{n \text{ times}}$$

A General Model for Multiplication

In early elementary school, students use “×” to symbolize multiplication, as in $2 \times 4 = 8$. Because in algebra an x is used to denote a variable, students start using a dot or parentheses to denote multiplication later, as in $2 \cdot 4 = 8$ or $(2)(4) = 8$. With technology like calculators and cell phones, we use an asterisk to denote multiplication as in $2 * 4 = 8$. This symbolic notation, especially when it is not consistent, can cause confusion.

A General Model for Multiplication

Let’s now consider some related terminology. If $n \cdot a = b$, then n is called the **multiplier**, a is called the **multiplicand**, and b is called the **product**.

Furthermore, n and a can be said to be **factors** of b; b is a **multiple** of both n and a.

A General Model for Multiplication

Using boxes, we can represent a **general model for multiplication** (Figure 4.3):

n times *a* = the amount *a* added *n* times

That is, *a* is the value of whatever is in each box.

Figure 4.3

A General Model for Multiplication

The general model shown in Figure 4.3 works well for Problems 1 and 2. However, the last two problems do not fit the model as well. We can force-fit them, but repeated addition is not the essence of those models.

Therefore, we look for other models that can represent these contexts better.

Area Model for Multiplication

Area Model for Multiplication

The **area model for multiplication** is an important one that we will see throughout the book. Figure 4.4 is a discrete representation for 4 times 3.

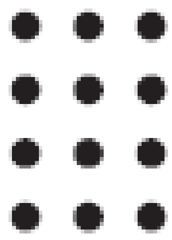


Figure 4.4

Area Model for Multiplication

If we change each dot to a square, as in Figure 4.5, many students can better see the connection between the repeated addition model and the area model.

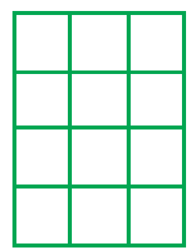


Figure 4.5

Cartesian Product Model for Multiplication

Cartesian Product Model for Multiplication

As we saw in Problem 4, the Cartesian product of any two sets *A* and *B* is the set consisting of *all* possible ways of combining elements of the first set with elements of the second set.

Using more formal language, we say that the Cartesian product of any two sets *A* and *B* consists of all possible ordered pairs such that the first element is from set *A* and the second element is from set *B*.

Cartesian Product Model for Multiplication

In mathematical notation, we write

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

In Problem 4, the Cartesian product of the set *S* of skirts and the set *B* of blouses is

$$\{(S_1, B_1), (S_1, B_2), (S_1, B_3), (S_1, B_4), (S_2, B_1), (S_2, B_2),$$
$$(S_2, B_3), (S_2, B_4), (S_3, B_1), (S_3, B_2), (S_3, B_3), (S_3, B_4)\}$$

Cartesian Product Model for Multiplication

The definition of multiplication as Cartesian product is similar to our definition of addition using set language:

> If $n(A) = a$ represents the number of elements in set A, and if $n(B) = b$ represents the number of elements in set B, then the number of elements in the Cartesian product of sets A and B is equal to the product of a and b.

Cartesian Product Model for Multiplication

In notation,

$a \cdot b = n(A \times B)$

That is, the value of *a* times *b* is equal to the number of elements in the Cartesian product of set *A* and set *B*.

Multiplication with Number Lines

Multiplication with Number Lines

Representing multiplication problems on a number line also highlights the context of multiplication as repeated addition. What multiplication problem is represented on the number line below (Figure 4.6)?

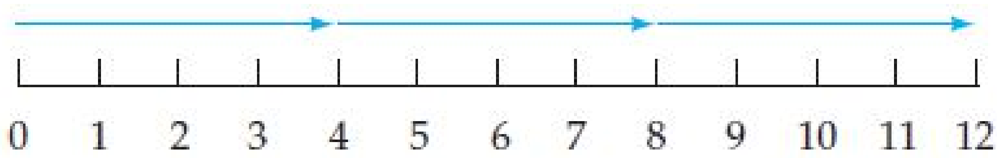


Figure 4.6

We can see the length of 4 represented 3 times, so this number line represents 3 × 4 = 12.

Properties of Multiplication

Properties of Multiplication

Let's say two children have birthday parties. One child has 7 party bags with 8 candies in each, and the other has 8 party bags with 7 candies in each. Numerically, both problems yield 56 candies, but the two problems don't look alike to many children. They are different situations.

Which of the properties that hold for addition also hold for multiplication? How can we use the four models of multiplication discussed earlier in this section to help us see these properties?

Properties of Multiplication

When adding, we found that "zero doesn't do anything." However, when we multiply any number by zero, the product is zero. This property of multiplication, which is not intuitively obvious to many children, is known as the **zero property of multiplication**:

$$a \cdot 0 = 0 \cdot a = 0$$

Properties of Multiplication

Many children are either intrigued or confused by the fact that when adding, “zero doesn’t change your answer,” but that when multiplying, “one doesn’t change your answer.” Stated in everyday language, “when we multiply any number by 1, we get the same number.” This is called the **identity property of multiplication**:

$$a \cdot 1 = 1 \cdot a = a$$

Properties of Multiplication

Just as with addition, when we multiply any two numbers, we get the same product regardless of the order; that is, $a \cdot b = b \cdot a$. This property is known as the **commutative property of multiplication**. While all the models for multiplication can be used to show the commutative property of multiplication, perhaps the area model is the most intuitive since all you have to do is turn the rectangle sideways. For example, a 3×4 rectangle could be turned sideways to show a 4×3 rectangle, and the two rectangles are the same size.

Properties of Multiplication

Just as with addition, grouping doesn't matter when we multiply several numbers. For example, $(5 \cdot 4) \cdot 7 = 5 \cdot (4 \cdot 7) = 140$, although the problem is much easier to do mentally in the first way. This grouping property is known as the **associative property of multiplication**:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

Properties of Multiplication

The **distributive property** connects multiplication to the operations of addition and subtraction. Consider the following problem given to third-grade students: How many cans of soda are in three 24-can cases?

Some will simply add 24 + 24 + 24.

Properties of Multiplication

Others will represent the problem with base ten blocks, as in Figure 4.7. They will count 6 tens and 12 ones, convert to 7 tens and 2 ones, and give the answer of 72.

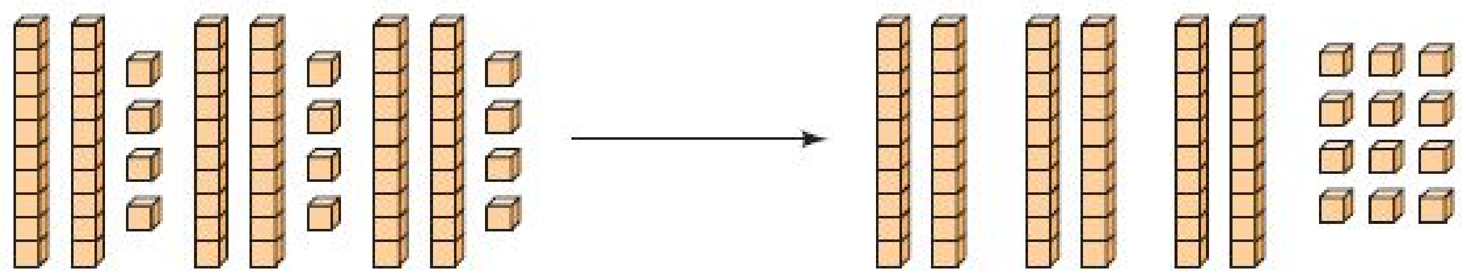


Figure 4.7

Properties of Multiplication

In this case, the students are using the **distributive property of multiplication over addition**:

$a(b + c) = ab + ac$

They have transformed $3 \cdot 24$ into $3 \cdot (20 + 4) = 3 \cdot 20 + 3 \cdot 4 = 60 + 12 = 72$ (Figure 4.8).

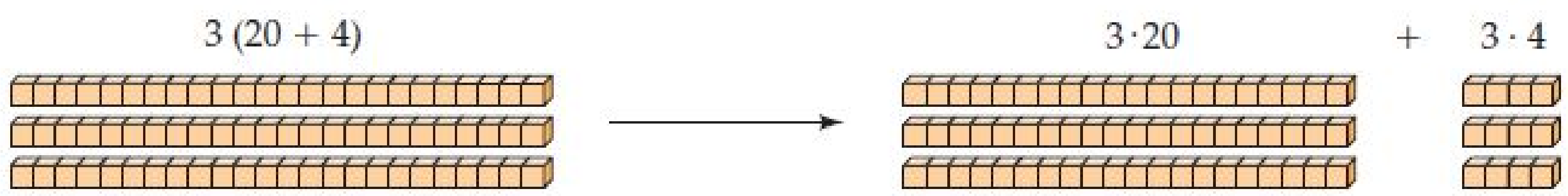


Figure 4.8

Properties of Multiplication

Others will solve the problem with money: 24¢ is 1 penny less than a quarter; do this 3 times and you will have 3 quarters take away 3 pennies—that is, $75 - 3 = 72$. In this case, the students are using the **distributive property of multiplication over subtraction**:

$$a(b - c) = ab - ac$$

The students have transformed $3 \cdot 24$ into

$$3 \cdot (25 - 1) = 3 \cdot 25 - 3 \cdot 1 = 75 - 3 = 72.$$

Changes in Units

Changes in Units

There is an important difference between multiplication and addition (or subtraction).

When we add or subtract two amounts, the units do not change: oranges plus oranges = oranges. However, this is not true with multiplication.

Changes in Units

Look back to the four multiplication problems presented at the beginning of this section. Do you see this?

- We are multiplying 3 *pieces* by 4 *cents per piece* and getting 12 *cents*.
- We are multiplying 4 *miles per hour* by 3 *hours* and getting 12 *miles*.

Changes in Units

- We are multiplying 4 *feet* by 3 *feet* and getting 12 *square feet*.
- We are multiplying 4 *blouses* by 3 *skirts* and getting 12 *outfits*.

The Multiplication Table

The Multiplication Table

Let us further examine the multiplication table to see how patterns can help children learn the table's 100 "multiplication facts." What patterns do you see in Table 4.1?

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Table 4.1

Investigation 4.1a – *A Pattern in the Multiplication Table*

A student noted that every odd number in the table is "surrounded" by even numbers but that the reverse is not the case. Why do you think every odd number in the table is surrounded by even numbers?

Investigation 4.1a – *Discussion*

A key to unlocking this mystery comes from the following generalizations: The product of two even numbers is always an even number, the product of two odd numbers is always an odd number, and the product of an odd number and an even number is always an even number. If you didn't make much progress on this question before, try to use this information and see whether you can explain this phenomenon now.

Investigation 4.1a – *Discussion*

continued

Let us look first at a specific case and then at a general case. The diagram below shows 35 (from 7 · 5) and the numbers that surround it in the table.

		6	7	8
		.	.	.
		.	.	.
		.	.	.
4	. . .	24	28	32
5	. . .	30	35	40
6	. . .	36	42	48

Investigation 4.1a – *Discussion* continued

More generally, we can write

		Even	Odd	Even
		⋮	⋮	⋮
Even	. . .	Even	Even	Even
Odd	. . .	Even	Odd	Even
Even	. . .	Even	Even	Even

Investigation 4.1a – *Discussion* continued

Thus, we see that each of the eight numbers surrounding an odd number in a multiplication table is either the product of two even numbers or the product of an odd number and an even number.

Thus, none of them can be an odd number.

Investigation 4.1a – *Discussion* continued

Using this pattern

We will present two ways in which this knowledge can help with multiplication facts.

First, a brief reflection on Investigation 4.1a yields the deduction that $\frac{3}{4}$ of the multiplication facts are even numbers; that is, only one in four multiplication facts is an odd number.

Investigation 4.1a – *Discussion* continued

Second, nowhere in the multiplication table do we have two odd numbers in a row.

Thus, for example, if we know that $7 \cdot 7$ is 49, then $8 \cdot 7$ can't be 55 or 57. The generalizations about the products of odd and even numbers would help, for example, a student who wasn't sure whether $7 \cdot 7$ was 48 or 49.

Investigation 4.1b – *Mental Multiplication*

Do the following computations in your head. Briefly note the strategies you used, and try to give names to them.

1. 64×5 **2.** 16×25 **3.** 15×12 **4.** 849×2 **5.** 60×30

As you read these strategies, once again monitor your own thinking. Do you understand how the strategies work? If not, do you simply need to reread the discussion, or would it be helpful to make up and do some similar problems, or would it be more helpful to practice with a friend?

Investigation 4.1b – *Discussion*

Here are some common strategies. You may have come up with others, as there are many ways to look at each problem.

In Problem 1 (64 × 5), use **multiples of 10** as a reference point:

$$64 \times 5 = \frac{1}{2} \text{ of } 64 \times 10 = \frac{1}{2} \text{ of } 640 = 320$$

Investigation 4.1b – *Discussion* continued

In Problem 2 (16×25), the **halve and double** method gives

$$16 \times 25 = 8 \times 50 = 400$$

Another way is to use the fact that $4 \times 25 = 100$ and break up the 16 as

$$16 \times 25 = 4 \times 4 \times 25 = 4 \times 100 = 400$$

Investigation 4.1b – *Discussion* continued

Problem 3 is interesting because it lends itself nicely to many different solution paths. It can be solved using the distributive property in two ways.

One way:

$15 \times 12 = 15 \times (10 + 2) = (15 \times 10) + (15 \times 2) = 150 + 30 = 180$

That is, 15×12 is seen as 12 groups of 15, which break apart into 10 groups of 15 + 2 groups of 15.

Investigation 4.1b – *Discussion* continued

Another way:

$15 \times 12 = 12 \times 15 = (12 \times 10) + (12 \times 5) = 120 + 60 = 180$

That is, 15×12 is seen as 15 groups of 12, which break apart into 10 groups of 12 and 5 groups of 12.

We can use the halve and double strategy:

$15 \times 12 = 30 \times 6 = 180$

Investigation 4.1b – *Discussion* continued

In Problem 4 (849×2), try compatible numbers in conjunction with the distributive property:

$$849 \times 2 = (850 - 1) \times 2 = 1700 - 2 = 1698$$

For Problem 5 (60×30), use multiples of 10:

$$60 \times 30 = 6 \times 3 \times 10 \times 10 = 18 \times 100$$

What people often think in this case is “18 with two zeros,” or 1800.

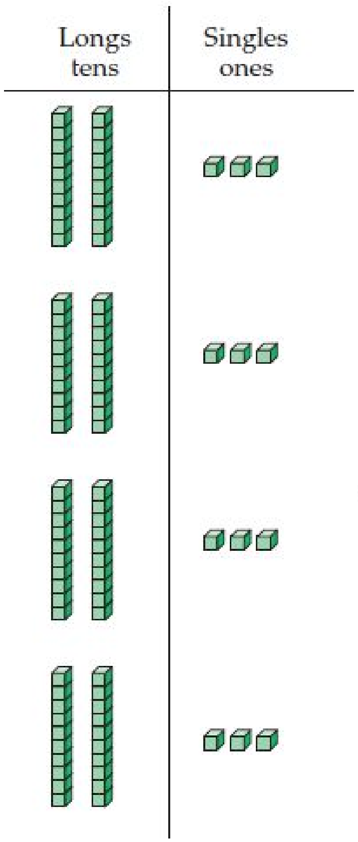
Understanding Multiplication with Larger Numbers

Understanding Multiplication with Larger Numbers

The big jump for children in learning how to multiply is to apply their knowledge of base ten and place value and to let go of strategies that work well for addition.

Understanding Multiplication with Larger Numbers

For example, 23 × 4 can be found in many ways, as shown below.



Add	Add pairs	Count the 10s Add the 3s	Take apart (1)	Take apart (2)	Take apart (3)
1 23 23 23 23 92	23 46 23 92 23 46 23	23 23 23 23 80 12 92	$(25 - 2) \times 4$ $25 \times 4 - 2 \times 4$ $100 - 8$	10×4 10×4 3×4 $40 + 40 + 12$	20×4 3×4 $80 + 12$

Understanding Multiplication with Larger Numbers

Some ways use base ten; some ways use powerful tools, and some ways do not.

Some of these ways, like the first three, help the children to see the close connection between addition and multiplication. The way at the far right illustrates a more powerful, efficient way to determine the answer.

Understanding Multiplication with Larger Numbers

One starting point for developing an understanding of the standard algorithm is to go beyond the basic multiplication facts, one step at a time. For example, how might you solve 16×7 if you didn’t know the algorithm?

When this problem is given to children, they solve it in many ways.

Understanding Multiplication with Larger Numbers

Children naturally invent the distributive property by applying multiplication as repeated addition to their multiplication facts. For example, 16×7 can be broken into $(8 \times 7) + (8 \times 7)$ or $(10 \times 7) + (6 \times 7)$.

We want the children to understand that breaking the numbers apart, using base ten, generally proves more powerful. Thus, this decomposition is ultimately more powerful: $16 \times 7 = (10 \times 7) + (6 \times 7)$. Toward that end, the problems become larger and larger.

Understanding Multiplication with Larger Numbers

For example, the children can construct the answer to 35 × 14 with the following string, where they begin with what they know and build up:

$3 \times 4 = 12$

$30 \times 4 = 120$ because they have studied the pattern when multiplying by 10

Understanding Multiplication with Larger Numbers

$5 \times 4 = 20$

$35 \times 4 = 140$ because 35×4 is equivalent to four 30s plus four 5s

$35 \times 10 = 350$ multiplying by10

$35 \times 14 = 490$ because 35×14 is equivalent to four 35s plus ten 35s

Understanding Multiplication with Larger Numbers

These problems, which are called string problems or cluster problems, help to lay the conceptual foundation for understanding the standard algorithm.

They also provide good practice in applying the distributive property, which is essential to understanding the algorithm.

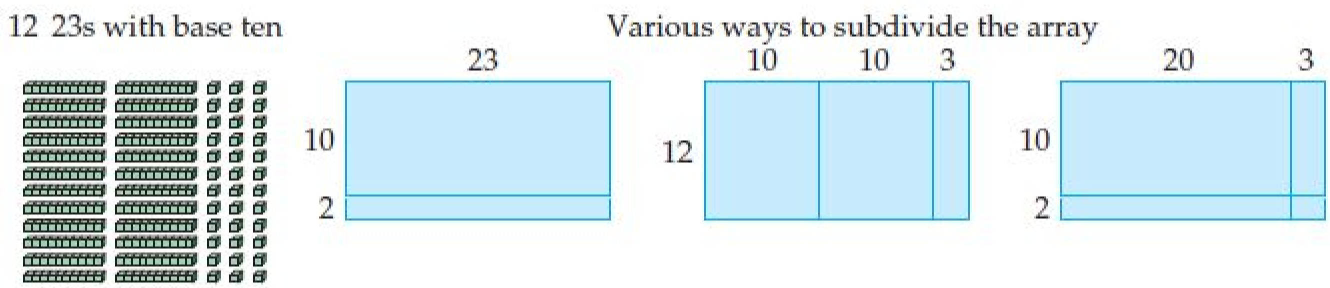
Understanding Multiplication with Larger Numbers

Another step in the development is to examine problems in context. For example, we can frame the problem 23 × 12 as: How much will it cost to blacktop a playground that measures 23 meters by 12 meters?

The natural representation of this problem, a rectangle, virtually *requires* the children to move toward the more powerful representation of multiplication as an array.

Understanding Multiplication with Larger Numbers

Below you can see various representations of the problem and then some of the solution paths that come out of these representations.



23×10	10×12	20×10
23×2	10×12	20×2
	3×12	3×10
		3×2
$230 + 46$	$120 + 120 + 36$	200
		40
		30
		+ 6
		276

Understanding Multiplication with Larger Numbers

All three rectangles employ the powerful, but often only partially understood, distributive property.

The first rectangle illustrates one way of cutting the rectangle apart. That is, twelve 23s can be broken into ten 23s and two 23s.

The second rectangle illustrates another way of cutting the rectangle apart. That is, twenty-three 12s can be broken into ten 12s plus ten more 12s plus three more 12s.

Understanding Multiplication with Larger Numbers

The third rectangle illustrates a way of cutting the rectangle apart that connects to the standard algorithm.

This way can be elicited by asking the children to fill the space with the least number of manipulatives. This can be done by filling the top left section with flats, the bottom right section with singles, and the other two sections with longs.

Understanding Multiplication with Larger Numbers

Investigations like these help teachers to see that although multiplication can be seen as repeated addition, if that is *all* you see multiplication as, then your students can achieve only limited understanding. Teachers also realize that we are still looking at parts and wholes. However, parts and wholes with addition and subtraction are conceptually much simpler.

For example, in adding two numbers (say 68 + 43), almost every way of taking apart the numbers will connect the closest place value (70 − 2 or 40 + 3).

Understanding Multiplication with Larger Numbers

However, when we are looking at multidigit multiplication, there are many ways to take apart the numbers, rather than just one or two ways.

The Multiplication Algorithm in Base Ten

The Multiplication Algorithm in Base Ten

We will now examine the standard multiplication algorithm, which is perhaps the most difficult of the algorithms to understand at the conceptual level.

It rests on an understanding that any multiplication problem can be represented as a rectangle and on an understanding of the distributive property. We will examine this algorithm with $56 \cdot 34$.

The Multiplication Algorithm in Base Ten

The rectangular model of the product

Without base ten, we are left with an imposing problem! Figure 4.9 shows 34 rows of 56 units.

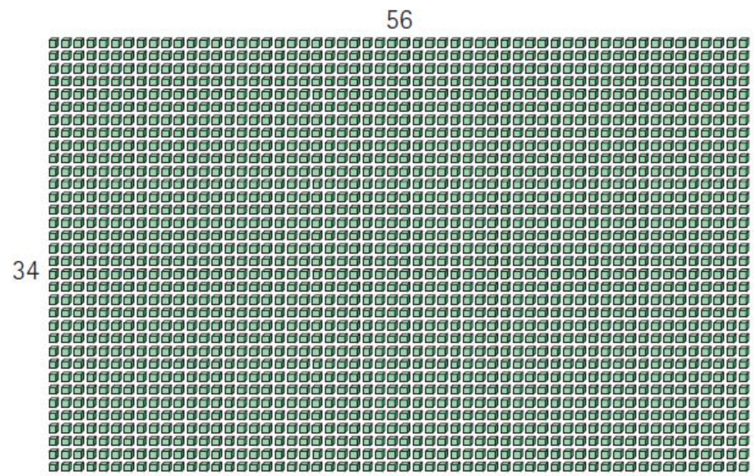


Figure 4.9

The Multiplication Algorithm in Base Ten

The answer is there, but who wants to count them all? However, if we apply our knowledge of base ten, we can find alternatives to counting each single unit.

Using base ten to model the product

One first step is to see what the problem would look like if we represented it using our knowledge of base ten (Figure 4.10).

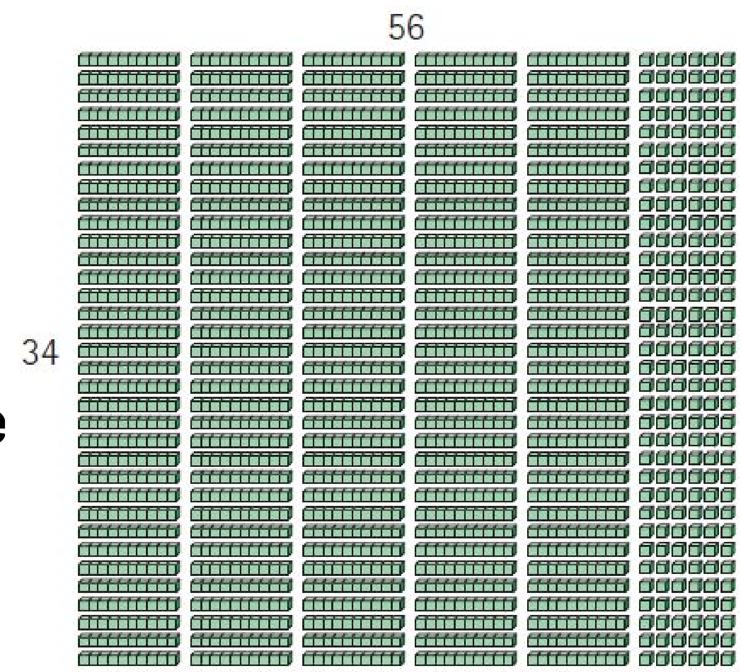


Figure 4.10

The Multiplication Algorithm in Base Ten

That is, we would have 56 (5 tens and 6 ones) 34 times. This makes our job only slightly easier. Who wants to add 56 thirty-four times?

Once again, our knowledge of base ten can make the counting process less tedious.

The Multiplication Algorithm in Base Ten

Using our knowledge that 10 ones make a ten, and that 10 tens make a hundred, we can turn units into longs and longs into flats (see Figure 4.11).

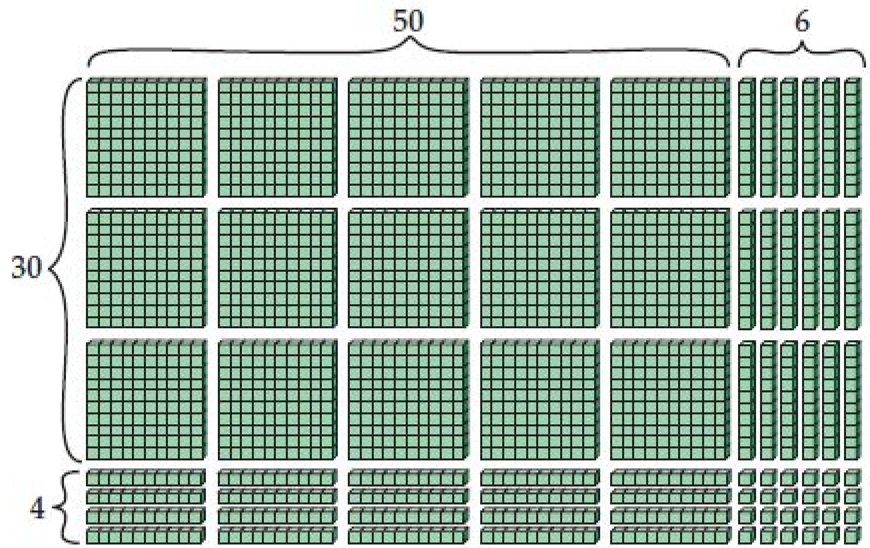


Figure 4.11

The Multiplication Algorithm in Base Ten

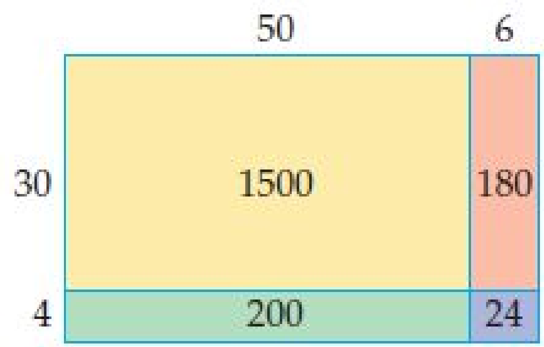
You can do this yourself by drawing a line after every 10 rows of 56s in Figure 4.10. Do you see the connection between these processes and the following diagram?

Do you see that this diagram is another representation of 56 × 34? Do you see connections between this diagram and how you would compute 56 × 34?

The Multiplication Algorithm in Base Ten

Counting the new groups

When we represent the problem using the rectangular model of multiplication, we see that the standard algorithm systematically does the regrouping for us. Let us first look at how we can quickly count the amount in each of the four regions in Figure 4.11.



The Multiplication Algorithm in Base Ten

- The units region consists of $4 \cdot 6 = 24$.
- The longs region to the left of the units region consists of $4 \cdot 50 = 20$ longs, which have a value of 200.
- The longs region above the units region consists of $30 \cdot 6 = 18$ longs, which have a value of 180.

The Multiplication Algorithm in Base Ten

- The flats region consists of 30 × 50 = 15 flats, which have a value of 1500.

Adding these four regions, we have

$$\begin{array}{r} 1500 \\ 180 \\ 200 \\ +\ 24 \\ \hline 1904 \end{array}$$

The Multiplication Algorithm in Base Ten

Thus, when we use the algorithm to determine the product $56 \cdot 34$, we determine four products (in this case, $4 \cdot 6$, $4 \cdot 50$, $30 \cdot 6$, and $30 \cdot 50$), and we know what to do with the products. These products are called **partial products**. When we represent the problem in expanded form, we can see how the four partial products connect to the algorithm.

$$
\begin{array}{r}
50 + 6 \\
\times 30 + 4 \\
\hline
200 + 24 \\
1500 + 180 \\
\hline
\end{array}
\qquad
\begin{array}{r}
56 \\
\times 34 \\
\hline
224 \\
+1680 \\
\hline
1904
\end{array}
$$

The Multiplication Algorithm in Base Ten

The standard multiplication algorithm

When we move up to larger multiplication problems, the manipulatives and visual representations become more cumbersome. However, if the students understand the process, they can apply that understanding to the larger problems. This understanding is also the foundation for skillful estimation.

For example, when we represent 324 times 6 in expanded form, we have three partial products.

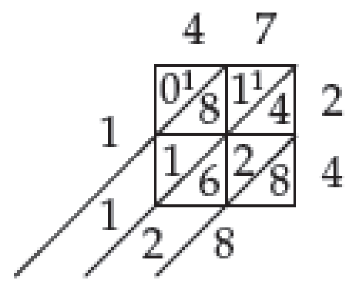
The Multiplication Algorithm in Base Ten

The value of the partial product 3 · 6 (which represents 300 times 6) is much larger than the value of the other two partial products (120 and 24). Therefore, 300 times 6 (1800) is a quick and reasonable estimate for this problem.

$$
\begin{array}{r}
324 \\
\times \quad 6 \\
\hline
\end{array}
\qquad
\begin{array}{lr}
 & 300 + 20 + 4 \\
\times & 6 \\
\hline
\multicolumn{2}{r}{1800 + 120 + 24}
\end{array}
$$

Investigation 4.1c – *An Alternative Algorithm*

The lattice algorithm that you learned for addition can be modified for multiplication. Observe the example below and see if you can figure out how it works and why it works.



Investigation 4.1c – *Discussion*

It works this way.

First, write the first number horizontally and the second number vertically.

Second, make a rectangle that has the same dimensions as your problem. In this case, since we have a 2-digit by 2-digit problem, we make a 2 × 2 rectangle.

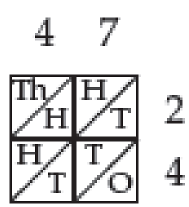
Third, as with lattice addition, draw diagonal lines through each square that extend below the square.

Investigation 4.1c – *Discussion* continued

Fourth, write the result of each partial product in the appropriate box.

Last, as with lattice addition, add diagonally. If the sum in any diagonal addition is greater than 10, trade to the next place value just as you do with the standard algorithm.

The "why" of this algorithm is very similar to the why of lattice addition. The diagram below shows the place value of each digit inside the lattice.



Th = Thousands
H = Hundreds
T = Tens
O = Ones

Investigation 4.1c – *Discussion* continued

As with lattice addition, the diagonals keep each digit with others of the same place value.

Investigation 4.1d – *Multiplication in Base Five*

Just as we determined that the base five addition table is one-fourth the size of the base ten addition table, so too with the base five multiplication table. We recommend that you take the time to make a base five multiplication table before reading on. It will make your ability to multiply, and later divide, in other bases much more meaningful and thus easier. Then perform the following computations in base five.

A. $32_{\text{five}} \times 3_{\text{five}}$ **B.** $34_{\text{five}} \times 31_{\text{five}}$

Investigation 4.1d – *Discussion*

You can use your fingers or manipulatives to make the base five multiplication table.

Base five	1	2	3	4	10
1	1	2	3	4	10
2	2	4	11	13	20
3	3	11	14	22	30
4	4	13	22	31	40
10	10	20	30	40	100

Investigation 4.1d – *Discussion* continued

A.

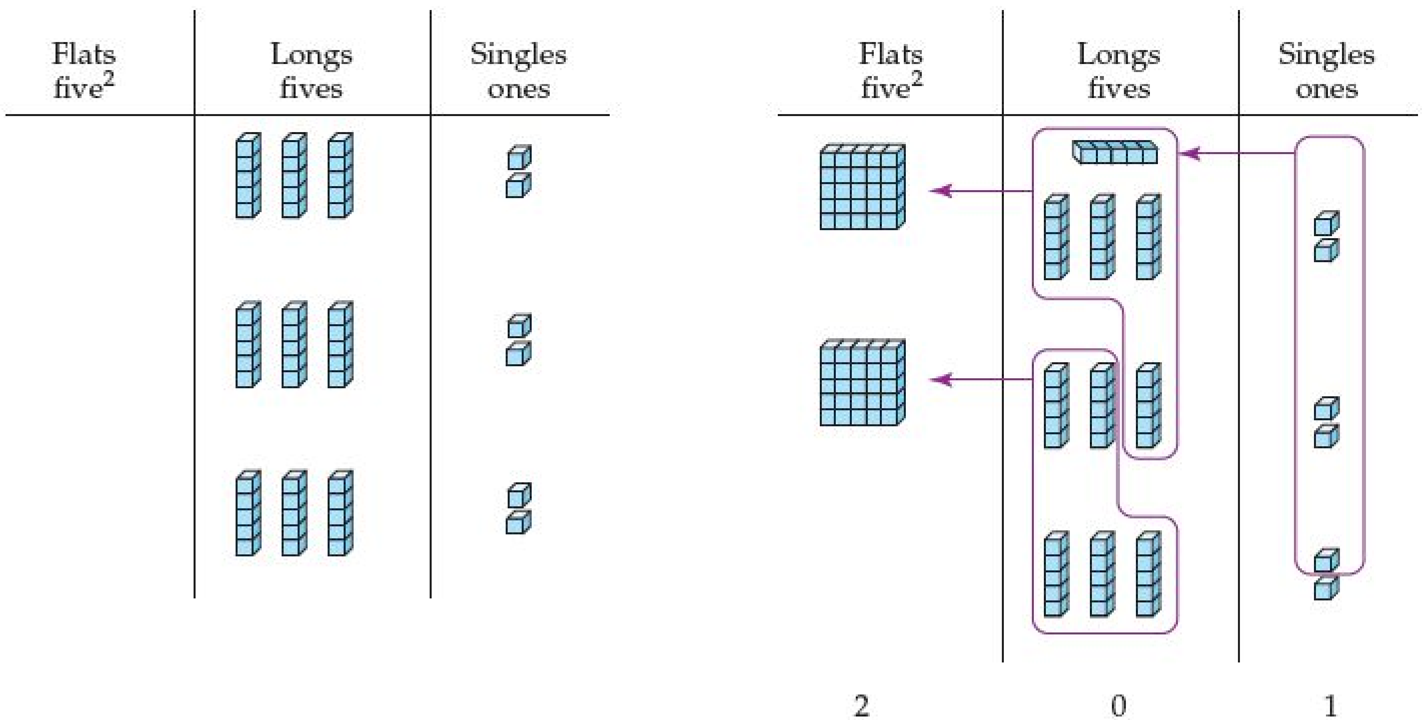
$$\begin{array}{r} \overset{1}{3}2_{\text{five}} \\ \times 3_{\text{five}} \\ \hline 201_{\text{five}} \end{array}$$

Because $2 \times 3 = 11_{\text{five}}$, place 1 in the singles place, and 1 above the 3 in the longs place.

$3 \times 3 = 14_{\text{five}}$. Add the regrouped 1 from the first partial product to the 14_{five} to get 20_{five}, which is placed appropriately below.

Investigation 4.1d – *Discussion* continued

Base blocks can help provide meaning to this regrouping as shown below.



Investigation 4.1d – *Discussion* continued

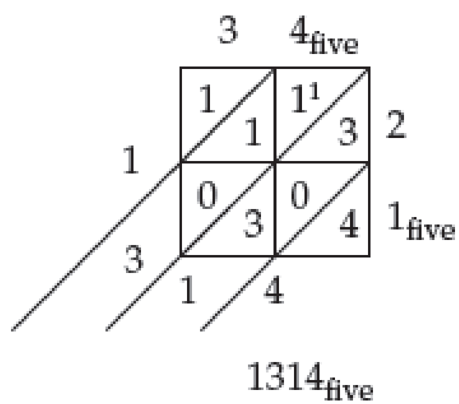
B.

$$\begin{array}{r} 34_{\text{five}} \\ \times\, 21_{\text{five}} \\ \hline 34\phantom{_{\text{five}}} \\ 123\phantom{0_{\text{five}}} \\ \hline 1314_{\text{five}} \end{array}$$

In this problem, there are four partial products: 4 × 1, 3 × 1, 4 × 3, and 3 × 3. The first row in the computation is 34 because there is no trading. In the second row, 4 × 2 = 13_{five}, place 3 in the longs place and trade for the 1 flat. Last, since 3 × 2 = 11_{five}, add the traded (from 4 × 2) to get 12_{five}. Place the 12 appropriately and then add the two rows to get the final answer.

Investigation 4.1d – *Discussion* continued

Some students find lattice multiplication easier especially with different bases. The key is to stay in the base, even when adding along the diagonals.



Investigation 4.1e – *Children's Mistakes in Multiplication*

In *Building a System of Tens*, a third-grade child multiplies 49 by 2 and gets 108. How do you think she got 108?

What misunderstanding(s) of place value and/or multiplication might be causing her problems?

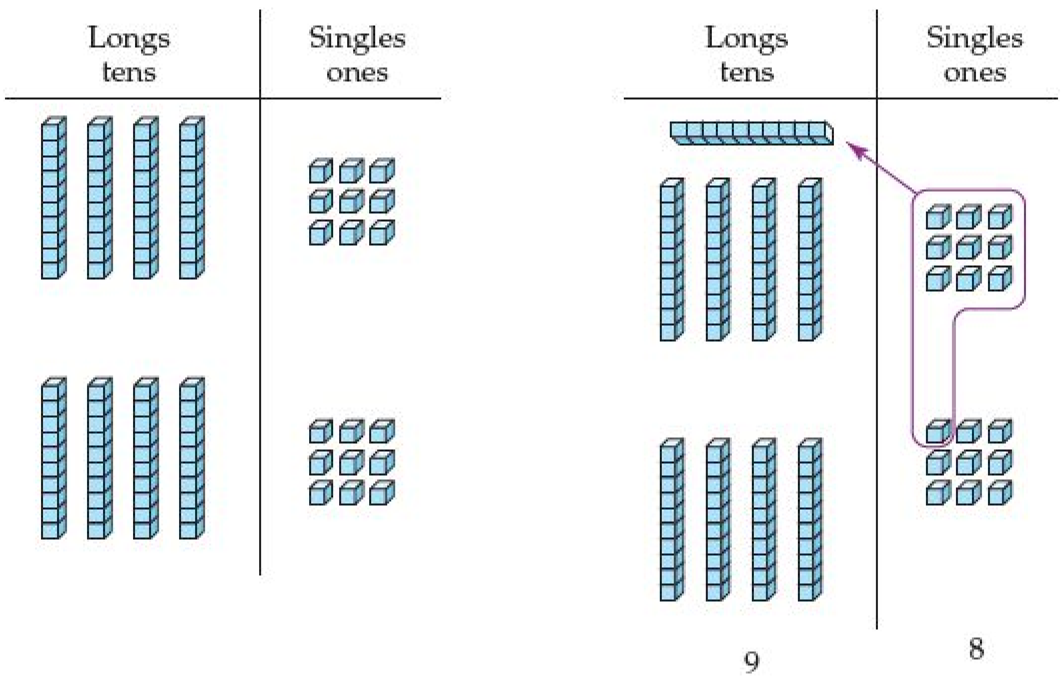
Investigation 4.1e – *Discussion*

$$\begin{array}{r} 49 \\ \times\ 2 \\ \hline 108 \end{array}$$

In this case, she correctly knew that 9 × 2 = 18. She placed the 8 in the ones place below and then traded 1 ten above the 4. It is likely that she generalized the addition algorithm in which you add the traded amount to the addition in the next place.

Investigation 4.1e – *Discussion* continued

Thus, she added the 1 to the 4 to make 5 and then multiplied 2 × 5. In multiplication, we need to do the multiplication first (4 × 2) and then add the traded 1 ten. Base ten blocks can help students see why this is as shown here.



Investigation 4.1f – *Developing Estimation Strategies for Multiplication*

In each of the problems below, obtain the best estimate.

A. Chip rode 78 miles last week. At this rate, how many miles will he ride this year?

Investigation 4.1f – *Discussion*

Strategy 1: Find a *lower bound* and an *upper bound* for the answer

Because we want to estimate the product of 78 and 52, if we round both numbers down to the nearest 10, we have

$70 \times 50 = 3500$.

Similarly, if we round both numbers up to the nearest 10, we have $80 \times 60 = 4800$.

Investigation 4.1f – *Discussion* continued

The actual product of 78 and 52 lies between these two numbers. The estimate from rounding down is called a **lower bound**, and the estimate from rounding up is called an **upper bound**.

Using this technique, we can quickly say that Chip will ride between 3500 and 4800 miles this year.

Investigation 4.1f – *Discussion* continued

Strategy 2: Round one number up and round the other number down

Thus, 78 × 52 becomes 80 × 50 = 4000.

Strategy 3: Use expanded form and estimate the sum of the four partial products

$$\begin{array}{r} 70 + \ \ 8 \\ \underline{50 + \ \ 2} \\ 140 + 16 \\ 3500 + 400 \qquad \end{array}$$

Investigation 4.1f – *Discussion* continued

One thought process might go like this: “70 × 50 = 3500, 50 × 8 is 400, and 70 × 2 is more than 100.

Thus, the answer will be 3500 plus more than 500, let’s say 4050.”

Determine the actual answer. How close were the different estimates?

B. Jane plans to start graduate school in September. She figures that she can save $345 per month for the next 9 months. How much will she have saved?

Investigation 4.1f – *Discussion*

Use the distributive property:

$$345 \times 9 = 345(10 - 1) = (345 \times 10) - (345 \times 1) = 3450 - 345$$

$$\approx 3100$$

C. There were 47,752 Americans killed or missing in Vietnam. The number of Americans killed or missing in World War II was about 6 times that number. Approximately how many Americans were killed or missing in World War II?

$$\begin{array}{r} 47{,}752 \\ \times \quad\quad 6 \\ \hline \end{array}$$

Investigation 4.1f – *Discussion*

You can get a rough estimate using rounding:

$50{,}000 \times 6 = 300{,}000$

You can get a more refined estimate by rounding and using mental math (double and halve):

$48 \times 6 = 96 \times 3 = 288$, or $288{,}000$

Investigation 4.1g – *Using Various Strategies in a Real-life Multiplication Situation*

A warehouse has 50 bays (places to stack pallets). Four pallets can be stacked in each bay, each pallet can hold 24 cartons, and each carton holds 12 boxes. How many boxes can the warehouse contain?

If each box sells for $4, what is the value of the merchandise in a full warehouse?

Investigation 4.1g – *Discussion*

As with many problems, there are several strategies that will lead to a solution.

Strategy 1: Draw a diagram

The diagram in Figure 4.12 is not “the right diagram” but rather an example of a useful diagram.

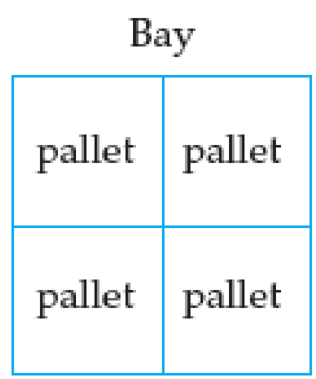


Figure 4.12

Investigation 4.1g – *Discussion* continued

If you didn't solve the problem or if you didn't solve it with a diagram, take a few moments to look at this diagram.

Does it help? How? Does it connect to or stimulate your thinking about what operation(s) might be involved?

Investigation 4.1g – *Discussion* continued

Strategy 2: Use smaller numbers

What if there were 3 bays, 4 pallets in each bay, 2 cartons in each pallet, and 5 boxes in each carton? Many researchers have found that if they give two problems that are mathematically identical, but one of which has big or messy numbers, the success rates can be dramatically different.

Do the smaller numbers help you to see the problem more clearly so that you can deduce what operation(s) to use?

Investigation 4.1g – *Discussion* continued

Strategy 3: Use dimensional analysis

Using dimensional analysis, you cancel the larger units to concentrate on the aspect of the problem you want to solve. In this problem, we have 50 bays.

However, we don’t want to know the amount in terms of bays, we want to know it in terms of a smaller unit—boxes.

Investigation 4.1g – *Discussion* continued

We can multiply 50 bays by 4 pallets per bay. Using dimensional analysis, this looks like

$$50 \cancel{\text{bays}} \cdot \frac{4 \text{ pallets}}{\cancel{\text{bay}}} = 200 \text{ pallets}$$

Because we do not want the answer in terms of pallets, we continue to change units:

$$200 \cancel{\text{pallets}} \cdot \frac{24 \text{ cartons}}{\cancel{\text{pallets}}} = 4800 \text{ cartons}$$

Investigation 4.1g – *Discussion* continued

Translating to an even smaller unit, we find that

$$4800\ \cancel{\text{cartons}} \cdot \frac{12\text{ boxes}}{\cancel{\text{carton}}} = 57{,}600\text{ boxes}$$

The value of the merchandise is

$$57{,}600\ \cancel{\text{boxes}} \cdot \frac{\$4}{\cancel{\text{box}}} = \$230{,}400$$

Investigation 4.1g – *Discussion* continued

A student more confident with using dimensional analysis can do the entire problem in one step:

$$50\ \cancel{\text{bays}} \cdot \frac{4\ \cancel{\text{pallets}}}{\cancel{\text{bay}}} \cdot \frac{24\ \cancel{\text{cartons}}}{\cancel{\text{pallet}}} \cdot \frac{12\ \cancel{\text{boxes}}}{\cancel{\text{carton}}} \cdot \frac{\$4}{\cancel{\text{box}}} = \$230,400$$

Investigation 4.1h – *Number Sense with Multiplication*

A. Without computing determine whether this number sentence is true or false.

$$68 \times 4 = (34 \times 2) + (34 \times 2)$$

Investigation 4.1h – *Discussion*

If we add 34 groups of 2 and 34 more groups of 2, we get 68 groups of 2, but 68 × 4 means 68 groups of 4. Therefore, this problem is false.

This is a common mistake made by children and adults, because both numbers at the right are half of the corresponding numbers at the left, so there is a sense of both being equal.

Investigation 4.1h – *Number Sense with Multiplication*

B. This problem is similar to puzzles archaeologists have to solve; the circles represent places where they cannot read the numbers. We have recovered these fragments of this multiplication problem. What was the problem, and what is the answer? What knowledge of multiplication can help you?

$$
\begin{array}{rcccc}
 & & & \bigcirc & 6 \\
\times & & & 4 & \bigcirc \\
\hline
 & & 3 & 2 & \bigcirc \\
 & \bigcirc & \bigcirc & \bigcirc & \\
\hline
 & \bigcirc & \bigcirc & \bigcirc & 4
\end{array}
$$

Investigation 4.1h – *Discussion*

This problem requires some analysis. Because the product has a 4 in the ones place, a first step is to ask 6 times what gives a 4 in the ones place? Many people stop with the 4 because 6 × 4 = 24.

However, if we are careful, we also see that 6 × 9 = 54. So now what do we do? What number in the tens place of the multiplicand will get us 324?

Investigation 4.1h – *Discussion* continued

That is 96 × 4, 86 × 4, 76 × 4, and so on. None of these work. What about 96 × 9? Way too big. Try smaller 46 × 9? 36 × 9? This works. Therefore, we have now solved part of the puzzle:

$$
\begin{array}{r}
36 \\
\times \;\; 49 \\
\hline
324 \\
\bigcirc\bigcirc\bigcirc \\
\hline
\bigcirc\bigcirc\bigcirc 4
\end{array}
$$

Actually, at this point, it is a simple matter to complete the problem to determine the rest of the missing places, and you can do this on your own.

Investigation 4.1h – *Number Sense with Multiplication*

C. Will the result of 43×87 be in the hundreds, thousands, or ten thousands? Explain your choice.

Investigation 4.1h – *Discussion*

If we use our estimation and rounding skills, we quickly see that $40 \times 90 = 3600$, so the answer is in the thousands.

Investigation 4.1h – *Number Sense with Multiplication*

D. Is this answer reasonable? Why or why not?

$$\begin{array}{r} 84 \\ \times 66 \\ \hline 4824 \end{array}$$

Investigation 4.1h – *Discussion*

Once we realize that there are four partial products, we can quickly resist the intuitive appeal of $4 \times 6 = 24$ and $8 \times 6 = 48$ to see the answer is larger than 4824.

Investigation 4.1i – *Patterns in Multiplying by 11*

Let us investigate what happens when we multiply a number by 11. From examining the first three problems in Table 4.2, can you predict the answer to 53 × 11? Make your prediction and then multiply 53 × 11 to check your prediction.

The problem	The product
26×11	286
35×11	385
42×11	462
53×11	?
62×11	?
73×11	?
75×11	?

Table 4.2

Investigation 4.1i – *Discussion*

If you had trouble, the *algorithm* for multiplying a two-digit number by 11 is to add the two digits together, and that number is the middle digit of the product.

Test this out for 62 × 11. When we get to 73 × 11, the problem becomes more challenging because the sum of the two digits is more than 9. Can you modify your thinking to predict the product of 73 × 11?

Investigation 4.1i – *Discussion* continued

First, we know that the digit in the ones place will stay the same. Because 3 + 7 = 10, the digit in the tens place will be a zero and the digit in the hundreds place increases by 1, like “trading.” This is how we get the answer of 803. Using this analysis, predict the product of 75 × 11.

We find that the algorithm works: 7 + 5 = 12. We still have 5 in the ones place, 2 in the tens place, and the hundreds place is now 1 more, 8, giving us the predicted product of 825.

Investigation 4.1i – *Discussion* continued

In a classroom, this is when we see who has really engaged. Has the problem gotten under their skin? Some people now will ask “what-if” questions: What if we are multiplying by a three-digit number? What if we multiply by 111? These will be left as exercises, if you can wait!

You can make up your own problems with a spreadsheet. The directions below work on Excel.

> First, type 26 in cell A1, then type 11 in cell A2. In cell A3, type = A1 * 11 and 286 will appear.

Investigation 4.1i – *Discussion* continued

To do a sequence of problems, like we have done, type this in cell A2: = A1 + 1.

In cell B2, type = B1.

Now, highlight cell C1 and drag down 1 space so that C2 is also highlighted.

Then press Control D. The answer to 27 × 11 will appear.

Now highlight cells A2, B2, and C2 all together. Drag down as far as you want to highlight more rows. Then press Control D!!!