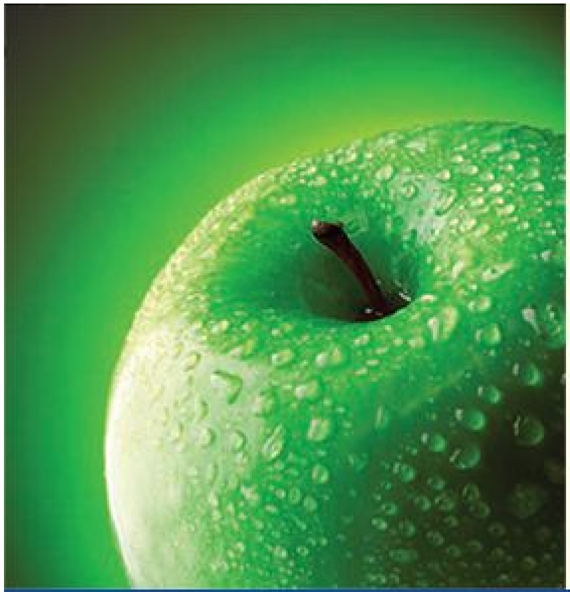


2 The Number System

SECTION 2.2

Fractions

What do you think?

- How would you define *fraction*?
- What pictures, models, and/or words explain why $\frac{1}{5}$ is less than $\frac{1}{3}$?
- What is the difference between the whole and the unit?

Fractions

The origin of the word *fraction* is interesting; it is derived from the Latin word *fractio*, which comes from the Latin word *frangere*, meaning “to break.”

In early American arithmetic books, the term *broken numbers* was often used instead of the word *fraction*. The first known mention of the actual word *fraction* was by Chaucer in 1321.

Fractions

Fractions are different from counting numbers and integers in a significant way: *two* numbers are needed to represent one amount! From another perspective, when we move from working with counting numbers to fractions, we are changing the question from *how many* to *how much*.

For example, we use counting numbers to count *how many* (such as 200) students go to college. We use fractions to quantify *how much* (such as $\frac{2}{3}$) of a particular graduating class goes to college.

Fractions

A counting number counts the number of units; a fraction tells us how much of a whole there is. This is a very important concept that we will explore more.

Clarifying Two Terms: Fractions and Rational Numbers

Clarifying Two Terms: Fractions and Rational Numbers

A **rational number** is a number whose value can be expressed as the quotient or ratio of two *integers a* and *b*, represented as $\frac{a}{b}$, where $b \neq 0$.

A **fraction** is a number whose value can be expressed as the quotient or ratio of *any two numbers a* and *b*, represented as $\frac{a}{b}$, where $b \neq 0$. For example, $\frac{\sqrt{2}}{3}$ is a fraction but not a rational number.

In either case, *a* is called the **numerator** and *b* is called the **denominator**.

Clarifying Two Terms: Fractions and Rational Numbers

Technically, the set of rational numbers is a subset of the set of fractions, because fractions include amounts like $\frac{\sqrt{2}}{2}$ and $\frac{\pi}{6}$, which are not rational numbers because the numerators are not integers.

In elementary school, children work with fractions that are rational numbers, and these will be our primary focus in this chapter. Therefore, we will generally use the term *fraction*.

Fractions in History

Fractions in History

The notion of $\frac{3}{4}$ may make sense to you, but it is not easy for many children, and it is such an abstraction that it is relatively recent in human history.

The Egyptians expressed all fractions (with the exception of $\frac{2}{3}$) as unit fractions—that is, fractions whose numerator is 1. They used the symbol , which they placed above a numeral to indicate a fraction. Thus, $\frac{1}{12}$ was written as .

The Egyptians' decision to represent fractional amounts using only unit fractions was a consequence of their difficulty with using two numbers to represent a single amount.

Fractions in History

As we saw earlier, the idea of representing *all* amounts with whole numbers was very appealing to the ancient Greeks, and so they did not even consider the idea of creating numbers that were not whole numbers.

Rather, they worked with ratios. For example, instead of saying that $\frac{2}{5}$ of the students at a college are male, they said that the ratio of males to females is 2 to 3.

Fractions in History

The Romans also avoided fractions. Rather than dealing with parts of a unit, they created smaller units. Their word for twelfth was *unica*, which is where our words *ounce* and *inch* come from.

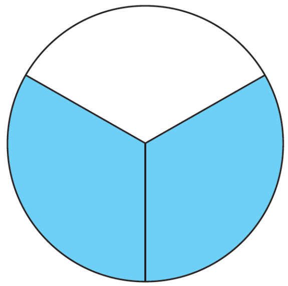
Our present method of writing fractions (for example, $\frac{2}{3}$) was probably invented by the Hindus. Brahmagupta (A.D. 628) wrote $\frac{2}{3}$. The bar seems to have been introduced by the Arabs.

Investigation 2.2a – *Tools for Defining a Unit Whole*

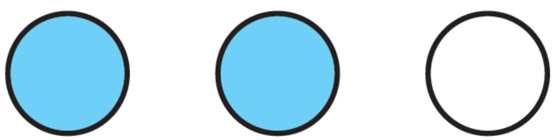
You are teaching a third-grade classroom and are going to draw a picture of $\frac{2}{3}$ on the board. When you ask your students how many circles to draw to illustrate $\frac{2}{3}$, you get two proposals. One student suggests that you start with one circle, another student suggests you start with three circles. Which student is correct?

Investigation 2.2a – *Discussion*

Each student is using a model that could be used to illustrate $\frac{2}{3}$. If one circle represents the unit whole, then $\frac{2}{3}$ would look like:



If three circles represent the unit whole, then $\frac{2}{3}$ would look like:



Investigation 2.2a – *Discussion* continued

Depending on the context of the problem, which model makes sense to the learner, and the teacher’s goals for the discussion in this situation, a strategic decision would be made as to which model to use or to use both models to gain a deeper understanding.

Using these pictures strategically illustrates using appropriate tools to solve a problem, as MP 5 suggests.

Investigation 2.2b – *Fraction Contexts with Visual Models: What Does $\frac{3}{4}$ Mean and Look Like?*

One of the interesting things about fractions is that we can represent them with different visual models, depending on the context. First draw as many different visual models of $\frac{3}{4}$ as you can. Then read on….

A. Explain how each of the visual models in Figure 2.6 represents $\frac{3}{4}$.

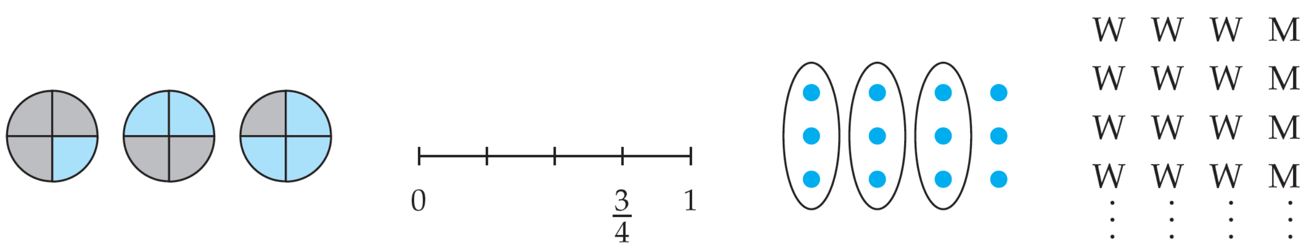


Figure 2.6

B. How are the following contexts for $\frac{3}{4}$ similar and different?

1. Four children want to share $\frac{3}{4}$ pies equally. How much pie does each child get?

2. Joey grew $\frac{3}{4}$ of an inch last month.

3. $\frac{3}{4}$ of a dozen donuts have been eaten.

4. At a college, $\frac{3}{4}$ of the students are women.

C. Which picture from A most closely matches each of the contexts from B?

Investigation 2.2b – *Discussion*

A key idea with fractions is the whole and the unit. We will discuss this idea throughout the next two sections, but introducing them now will help us to build our understanding. The unit refers to what equals 1. The whole is the given object or total amount.

1. Four children want to share 3 pies equally. How much pie does each child get?

This situation illustrates **fractions as a quotient**. We are dividing 3 pies (the whole) by 4 children.

Investigation 2.2b – *Discussion* continued

The answer of $\frac{3}{4}$ pie per child is each child's share. In other words, each child gets $\frac{3}{4}$ pie per child is each child's share. In other words, each child gets $\frac{3}{4}$ of a pie (the unit).

Here, 1 pie is the context for "1," but the whole, *or total amount we are working with*, is 3 pies. When we are dealing with a fraction $\frac{a}{b}$ in the context of a quotient, an amount *a* needs to be shared or divided equally into *b* groups.

The first visual model in part A relates to this context.

Investigation 2.2b – *Discussion* continued

Another way to look at this is in Figure 2.7.

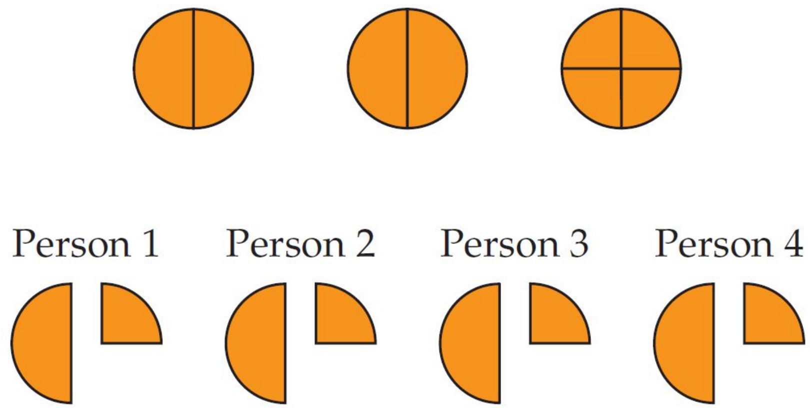


Figure 2.7

This type of visual model is called an **area model** since the size (or area) of the pies is the whole. The pieces also have to be the same size (or area).

Investigation 2.2b – *Discussion* continued

This is the earliest model of fractions that children encounter, even before school with real-life examples like $\frac{1}{2}$ of a sandwich (where the whole is the size of the sandwich). Along with circles, other area models include fraction rectangles, grid paper, and pattern blocks.

2. Joey grew $\frac{3}{4}$ of an inch last month.

This situation illustrates **fractions as a measure**. To measure the appropriate location of $\frac{3}{4}$, we must divide (partition) the unit length (1 inch) into 4 equal lengths.

Investigation 2.2b – *Discussion* continued

The length of 3 of those equal lengths shows how much Joey grew (Figure 2.8). Here, both the whole and the unit are 1 inch.

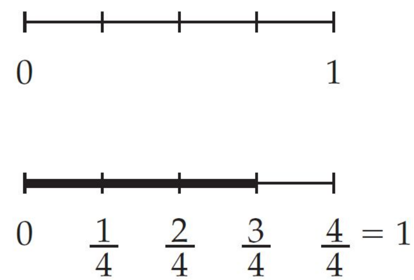


Figure 2.8

This type of visual model is called a **linear model** since the length of the line segment is the whole. The pieces of the line also have to be the same size (or length). Along with number lines, other linear models include Cuisenaire rods and Singapore bar models.

Investigation 2.2b – *Discussion* continued

3. $\frac{3}{4}$ of a dozen donuts have been eaten.

This situation illustrates **fractions as an operator** on the set of donuts, because we take 3 out of every 4 donuts in a box of 12 donuts.

In this case, the whole is 12 donuts (what we are taking $\frac{3}{4}$ of), and the unit is 1 dozen (because we use “1” to represent a dozen donuts). Taking $\frac{3}{4}$ of the 12 donuts is 9 donuts, which is $\frac{9}{12}$ of the dozen.

Investigation 2.2b – *Discussion* continued

This model helps us to understand equivalent fractions like $\frac{3}{4} = \frac{9}{12}$, which we will explore more in depth later in this section. Figure 2.9 shows the 9 out of 12 donuts, or $\frac{9}{12}$ of the dozen, which is equivalent to 3 columns out of 4 columns, or $\frac{3}{4}$ of the dozen.

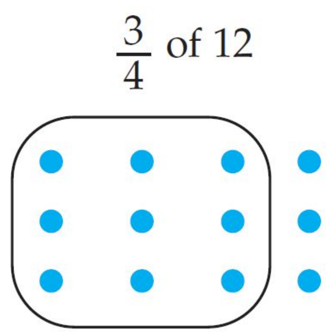


Figure 2.9

Investigation 2.2b – *Discussion* continued

This type of visual model is called a **set model** since the set of objects (the 12 donuts) is the whole. While the area and linear models are a size relationship (in other words the areas or lengths are divided into equal-size pieces), the set model is not a size relationship.

We can talk about $\frac{3}{4}$ of a set of animals or $\frac{3}{4}$ of a set of shapes. In these examples, the size of the objects is not relevant; the whole is the number of objects, not the size. Any collection of objects (blocks, candies, fruits, and so on) can be used as set models.

Investigation 2.2b – *Discussion* continued

4. At a college, $\frac{3}{4}$ of the students are women.

This situation illustrates **fractions as a ratio**. We do not know the total number of students in the college, which is the whole, but we do know that if we divided the total number of students into four groups with equal numbers of students in each, then the number of women would be three of those four groups.

Sometimes we might see this relationship in ratio notation: the ratio of women to total number of students is $3 : 4$.

Investigation 2.2b – *Discussion* continued

Figure 2.10 shows this relationship visually.

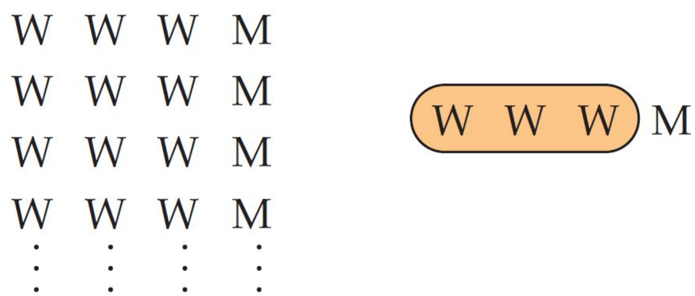


Figure 2.10

Because the whole is a set of students, this is related to a set model (even though in this situation we do not know how many are in that set). The whole is the total number of students.

Fractions in History

What do these contexts have in common?

There are certain important ideas that are in all four contexts:

- Each context can be interpreted in part–whole relationships. In each case a whole has been divided into four parts.
- Something is to be partitioned into parts of equal size (value). The something can have a value of 1, in which case the unit = the whole. The something can have a value ≠ 1, in which case the unit ≠ the whole

Fractions in History

- The numerator and denominator are like codes that tell us about the relative sizes of the parts and the unit, and the code is multiplicative in nature. For example, when we say $\frac{1}{2}$, the relationship is that the value of the denominator is twice the value of the numerator. Thus, $\frac{1}{2}$ has the same value as $\frac{4}{8}$, not $\frac{7}{8}$.

  Because in both $\frac{1}{2}$ and $\frac{4}{8}$, the value of the denominator is twice the value of the numerator.

Fractions in History

The unit and the whole are not always the same!

Equating the unit with the whole is one of the most common misconceptions that people have about working with fractions.

Look back on the four problems posed in connection with Figure 2.16 and consider the question: What do we mean by “the whole” and by “the unit”?

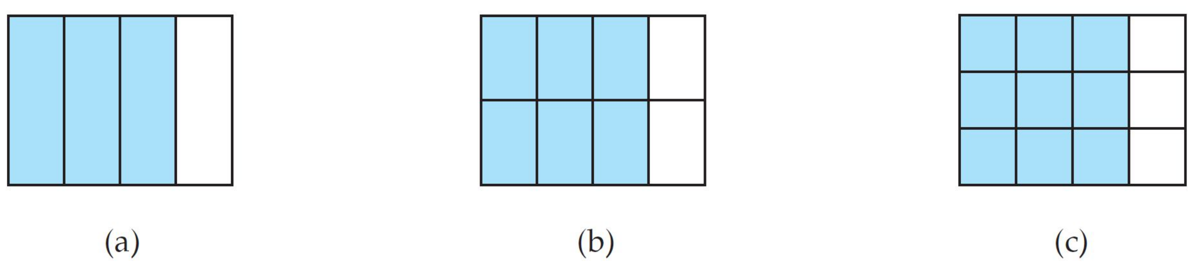


Figure 2.16

Fractions in History

The whole is the given object or amount. The unit is that amount to which we give a value of 1 : 1 inch, 1 pie, 1 person. In some cases, the whole and the unit are the same. For example, if Lisa gets $\frac{1}{2}$ of a pizza and Liam gets $\frac{1}{3}$ of the pizza, the whole is 1 pizza, and the unit is also 1 pizza.

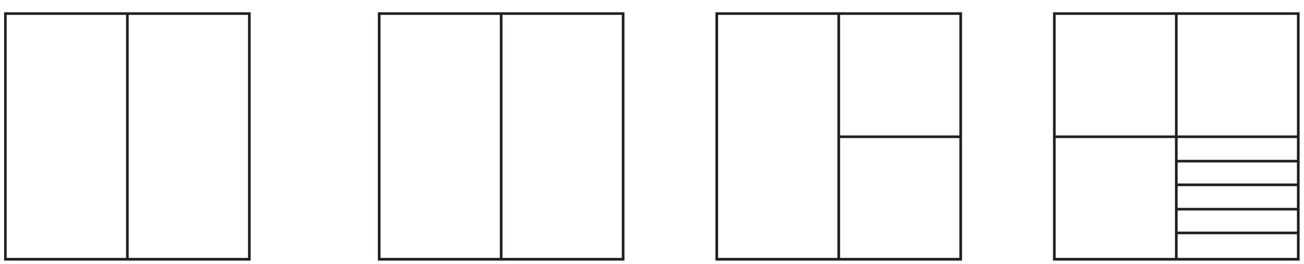
However, if 3 pizzas are divided among 4 people, then the whole is 3 pizzas but the unit is 1 pizza. Understanding the concept of units and wholes is a major key in understanding fractions.

Investigation 2.2c – *Wholes and Units: Sharing Brownies*

Five children need to share four brownies. How much does each child get? Solve this using visual and/or physical models.

Investigation 2.2c – *Discussion*

One way to do this is to divide each brownie into fifths and give each child four pieces. However, that is not satisfying to most children because they want bigger pieces. A common solution looks like the figure below. Most children have no trouble saying that each child gets $\frac{1}{2}$ of a brownie and $\frac{1}{4}$ of a brownie, but what is the name for the smallest piece that each child gets?



Investigation 2.2c – *Discussion* continued

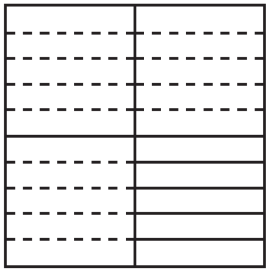
The smallest amount is $\frac{1}{20}$ of a brownie. Children will arrive at this conclusion in different ways. Some will partition the whole brownie so that all pieces are the same size, while others realize that this smallest piece is $\frac{1}{5}$ of $\frac{1}{4}$ of a brownie, and $\frac{1}{5}$ of $\frac{1}{4}$ is $\frac{1}{20}$.

When children incorrectly say that the smallest piece is $\frac{1}{5}$, they are seeing that last quarter of a brownie as the whole and that whole has been divided into 5 equal pieces.

Investigation 2.2c – *Discussion* continued

However, the value of the denominator of a fraction is connected to how many of those pieces it takes to make the unit (that which has a value of 1).

In this case, it takes 20 of those small pieces to make 1 brownie, and thus the size of that smallest piece is $\frac{1}{20}$ (of 1 brownie). We use an area model (dividing the area of the rectangles) to represent the brownies.



Fractions in History

With whole numbers, counting is simpler. A can of soda is 1 can; a six-pack of soda is 6 cans. We can count by 1 or by 6, for example, 24 cans or 4 six-packs. With fractions, we are talking about parts of units, and the fraction that our amount represents varies depending on the unit. For example, 6 inches is 6 inches.

However, if our unit is a foot, 6 inches is $\frac{1}{2}$ of a foot, but if our unit is a yard, 6 inches is $\frac{1}{6}$ of a yard. Becoming comfortable with unitizing is an important part of developing an understanding of fractions.

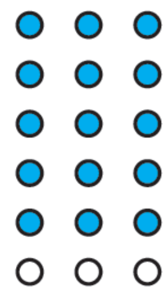
Fractions in History

Below are other examples of this concept.

Hour	4 hours is what part of a day?	4 hours is $\frac{1}{6}$ of a day.
	4 hours is what part of a work week?	4 hours is $\frac{1}{10}$ of a 40 hour work week.
Nickel	1 nickel is what part of a dime?	1 nickel is $\frac{1}{2}$ of a dime.
	1 nickel is what part of a dollar?	1 nickel is $\frac{1}{20}$ of a dollar.

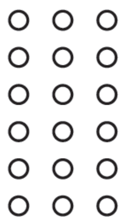
Investigation 2.2d – *Unitizing*

Please do the following problems as children who do not yet have algorithms might do them. For example, to determine $\frac{5}{6}$ of the circles we need to partition them into six equivalent groups. When we do this, $\frac{1}{6} = 3$ circles, and five groups of three circles represents $\frac{5}{6}$ of the whole.

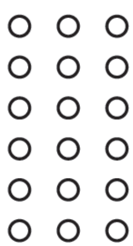


Investigation 2.2d – *Unitizing* continued

(a) Shade $\frac{2}{3}$ of the circles.

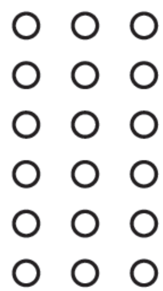


(b) Shade $\frac{4}{9}$ of the circles.



Investigation 2.2d – *Unitizing* continued

(c) Shade $\frac{7}{18}$ of the circles.



Investigation 2.2d – *Discussion*

The whole in each case is the set of 18 circles (which makes this a set model). The table and models below help show the answers. The second column shows the number of circles to shade.

The third column shows how many are in each group when we partition the set into the number of equal groups the denominator calls for.

For example, to make $\frac{5}{6}$, we must partition the circles into six equal groups, and we see that $\frac{1}{6}$ is equivalent to three circles.

Investigation 2.2d – *Discussion* continued

The fourth column shows us the equivalence of the original fraction and our shading. For example, $\frac{5}{6}$ is equivalent to 15 circles, which is equivalent to $\frac{15}{18}$ of the circles. Work on unitizing naturally develops understanding of equivalence, which we will explore very soon.

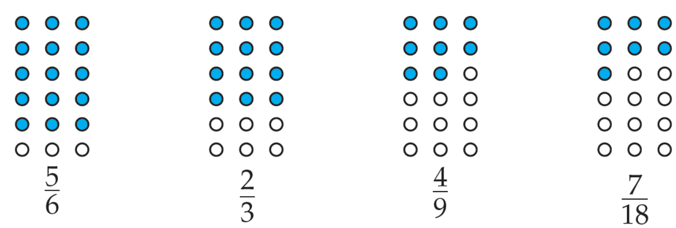
It is important to note that we could get the answers simply by using the algorithm:

$$\frac{5}{6} = \frac{5 \times 3}{6 \times 3} = \frac{15}{18}.$$

Investigation 2.2d – *Discussion* continued

Here, our goal is more than getting the answer; it is traveling the terrain that your future students will walk as they develop the ability to work confidently with fractions.

Fraction	Number of circles shaded	Unitizing	Equivalence
$\frac{5}{6}$	15	$\frac{1}{6} = 3$ circles	$\frac{5}{6} = \frac{15}{18}$
$\frac{2}{3}$	12	$\frac{1}{3} = 6$ circles	$\frac{2}{3} = \frac{12}{18}$
$\frac{4}{9}$	8	$\frac{1}{9} = 2$ circles	$\frac{4}{9} = \frac{8}{18}$
$\frac{7}{18}$	7	$\frac{1}{18} = 1$ circle	$\frac{7}{18} = \frac{7}{18}$



Investigation 2.2e – *Fundraising and Thermometers*

An organization has set \$300,000 as their fundraising goal. A large sign with a big thermometer sits by the street in front of the building..

Figure 2.11 shows their progress after 24 days (on the left) and after 49 days (on the right). Describe the progress of the fundraiser in as many ways as possible.

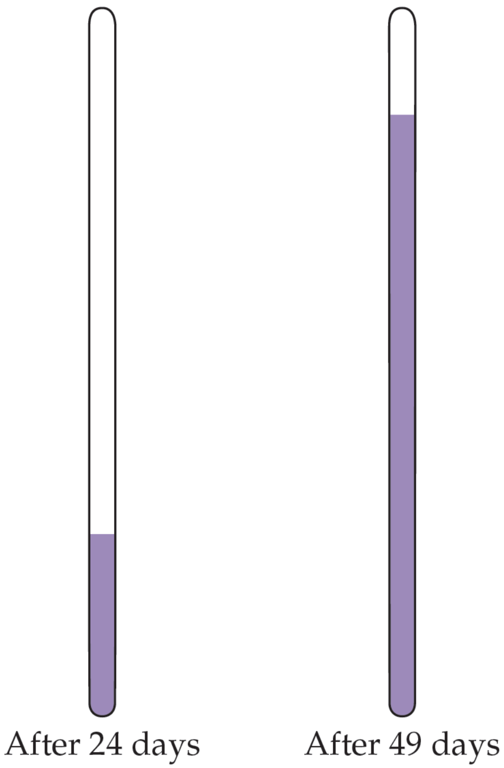


Figure 2.11

Investigation 2.2e – *Fundraising and Thermometers* continued

The notion of multiple representations and equivalence are in the foreground here and are essential for a deeper understanding of important mathematical ideas. Answer the following questions, each of which represents different interpretations and representations of the progress.

A. They are about $\frac{1}{4}$ of the way after 24 days, as shown on the thermometer on the left. Describe how one might arrive at this answer.

Investigation 2.2e – *Fundraising and Thermometers* continued

B. If the progress of the fundraiser continues at the rate it did during the first 24 days, how long will it take them to reach their goal?

C. Determine a fraction to represent their progress after 49 days.

D. About how many dollars have they raised after 49 days? Can you determine this answer mentally?

E. If progress of the fundraiser continues at the rate it did during the first 49 days, how long will it take them to reach their goal?

Investigation 2.2e – *Discussion*

A. The length of the thermometer (a linear measure) is the whole of this linear model. You could trace the picture on a piece of paper and fold. You could use the amount that is shaded as a ruler and see how many of these lengths fill the thermometer. In this case, you are using this length as your unit and it takes about four of that length to make the whole.

B. If they have made $\frac{1}{4}$ of their goal in 24 days, then they would make their goal in $4 \times 24 = 96$ days, or just over three months.

Investigation 2.2e – *Discussion* continued

C. We could use the nonshaded length as our unit. Realizing that it is $\frac{1}{6}$, we would conclude that the organization has reached $\frac{5}{6}$ of their goal after 49 days. We could also estimate thirds, then fourths, then fifths, and so on, until one division seems appropriate.

D. $\frac{5}{6}$ of \$300,000 is \$250,000. Here is one way to determine the amount mentally. First, we can see that $\frac{1}{6}$ of 300,000 is 50,000. Then we multiply 50,000 by 5 to get 250,000.

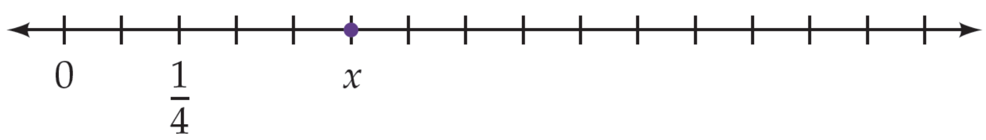
Investigation 2.2e – *Discussion* continued

E. If $\frac{5}{6}$ is equivalent to 49 days, then $\frac{1}{6}$ is equivalent to about 10 days ($\frac{1}{6}$ is $\frac{1}{5}$ of $\frac{5}{6}$, and $\frac{1}{5}$ of 49 is about 10). If $\frac{1}{6}$ is equivalent to about 10 days, then $\frac{6}{6}$ is equivalent to about 60 days.

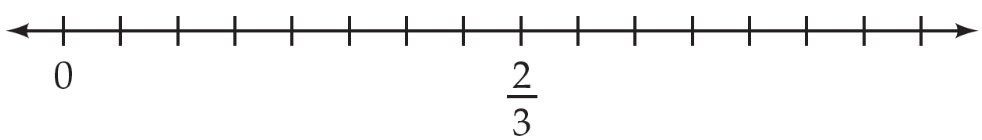
Investigation 2.2f – *Partitioning with Number Line Models*

A. Place $\frac{5}{6}$ on this number line.

B. Determine the value of *x* on the number line.



C. Locate $\frac{5}{6}$ on the number line.



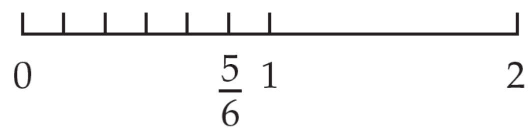
Investigation 2.2f – *Discussion*

A. This question requires you to grapple with the difference between the unit and the whole. In this case, the whole and the unit are not the same. This is important, because the meaning of the denominator *is in relation to the unit*.

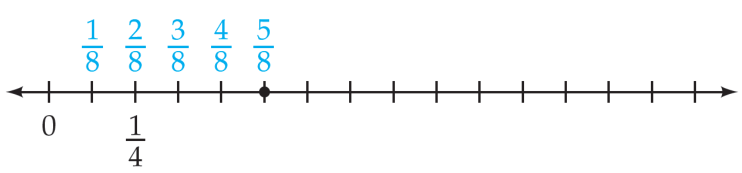
That is, to find the location of $\frac{5}{6}$, we do not take the whole line and divide it into six equal lengths. Rather, we first must determine the unit length and then divide that length into six equal lengths.

Investigation 2.2f – *Discussion* continued

This difference between the whole and the unit cannot be overemphasized.

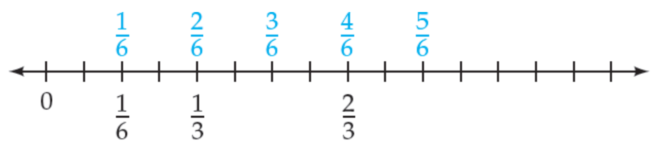


B. We can deduce that if $\frac{1}{4}$ represents two partitions from zero, then one partition from zero would be $\frac{1}{8}$ (which is $\frac{1}{2}$ of $\frac{1}{4}$). Now, we can see the location of $\frac{5}{8}$.



Investigation 2.2f – *Discussion* continued

C. In this case, $\frac{2}{3}$ represents eight partitions to the right of zero. Thus, four partitions would have a value of $\frac{1}{3}$, and two partitions would have a value of $\frac{1}{6}$. So the location of $\frac{5}{6}$ is 5 × 2 = 10 partitions to the right of zero.

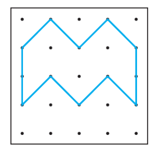


As you will discover when you teach children, it is important to explore concepts with different models. Now we will investigate fractions with area models, which are often encountered in elementary schools with Geoboards, fraction circles or rectangles, and pattern blocks.

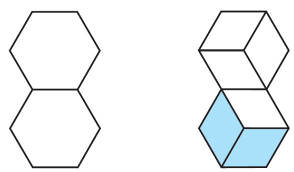
Investigation 2.2g – *Partitioning with Area Models*

A. If  $= 1$, show $\frac{4}{5}$.

B. Determine what fraction of the Geoboard is covered.

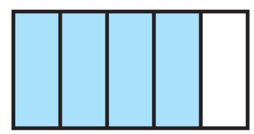


C. If the area of the two hexagons equals one unit, what fraction does the area of the two blue parallelograms equal?



Investigation 2.2g – *Discussion*

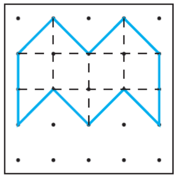
A. Divide the rectangle into five equal regions, and shade in four of them. The area of these four regions $= \frac{4}{5}$ of the area of the whole rectangle.



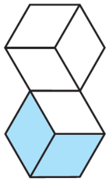
B. In the first case, the value of the entire Geoboard is 16 (unit squares). We can decompose the polygon into squares and triangles whose value is clearly $\frac{1}{2}$ (square).

Investigation 2.2g – *Discussion* continued

When we count the squares and $\frac{1}{2}$ squares, we have a value of 8, and thus the shape covers $\frac{8}{16}$ or $\frac{1}{2}$ of the area of the Geoboard.

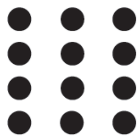


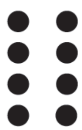
C. The lines help us to see that two of the blue parallelograms cover $\frac{2}{6}$ or $\frac{1}{3}$ of the area of the two hexagons.



Finally, let us investigate fractions with set models.

Investigation 2.2h – *Partitioning with Set Models*

A. If  $= 1,$ show $\frac{5}{6}.$

B. If  $= \frac{4}{3},$ show 1.

Investigation 2.2h – *Discussion*

A. To make sense of this question, we must partition the dots into six equal-size groups. Recall the partitioning model of whole-number division.

We then take five of those groups to show $\frac{5}{6}$ [Figure 2.12].

Figure 2.12a

Investigation 2.2h – *Discussion* continued

B. There are different ways to make sense of this situation and answer the question. For example, we can focus on the numerator, which indicates that we have four equal parts. Thus, each part—that is, $\frac{1}{3}$ —contains two dots. When we then focus on the denominator, we find that three of these equal parts represent the unit (that is, three of those parts have a value of one).

Figure 2.12b

Investigation 2.2i – *Determining an Appropriate Representation*

Jose paid \$12 for a box of chocolates that weighed $\frac{3}{4}$ pound. What is the price of one pound (at this rate)?

Investigation 2.2i – *Discussion*

An area model is one possible representation, because boxes of chocolates are often rectangular in shape. If we look at the box in terms of weight, we have $\frac{3}{4}$ of a pound. If we look at the box in terms of money, it costs \$12. In one sense, we are saying that $\frac{3}{4}$ of a pound is equivalent to \$12.

Thinking of the problem as "we have three parts, which are equivalent to \$12," the model helps us see that each part (that is, each quarter-pound) has a value of \$4 (Figure 2.13), and so one pound will cost \$16.

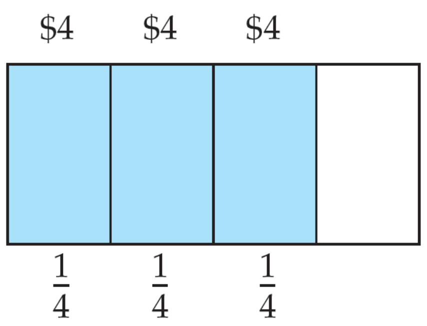


Figure 2.13

Equivalent Fractions

Equivalent Fractions

Having opened the concept of equivalent fractions in Investigation 2.2d, let us now examine the concept of equivalent fractions.

Equivalence

- What does *equivalent* mean?
- Where else have you encountered the notion of equivalence in mathematics and in life outside school?

Equivalent Fractions

One way of looking at “equivalent” comes from taking apart the actual word: *equivalent*, or equal value. We use this notion with regrouping whole numbers; for example, 1 ten and 2 ones is equivalent to 12 ones. You use equivalence with money every day; for example, one quarter is equivalent to 25 pennies.

Two fractions are **equivalent fractions** if they have the same value.

Equivalent Fractions

Using models

Can you illustrate the equivalence of $\frac{3}{4}$ and $\frac{6}{8}$ using one or more of the fraction models we have discussed: area, length, or set?

We can use the set model to illustrate equivalence. For example, consider a set of 8 dots [Figure 2.14(a)].

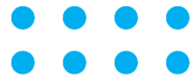


Figure 2.14(a)

Equivalent Fractions

If we take 6 of them, we literally have $\frac{6}{8}$ [Figure 2.14(b)].

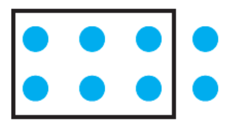


Figure 2.14(b)

However, we can also partition this set of 8 dots into 4 equal groups of 2 dots; if we then take 3 of these 4 equal groups, we have, by definition, taken $\frac{3}{4}$ of the set [Figure 2.14(c)].

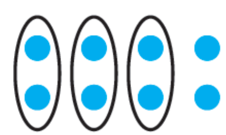


Figure 2.14(c)

Equivalent Fractions

Take a piece of paper and fold it in half. Then fold it in half again. [*Note:* There are several ways to do this.] Shade 3 of the 4 regions to show $\frac{3}{4}$. Now fold the paper in half again. Now we see that the paper has been folded into 8 regions and that 6 are shaded.

Thus, $\frac{3}{4} = \frac{6}{8}$: that is, these two fractions represent the same *part* of the whole. This is an area model since $\frac{3}{4}$ of the area is the same size as $\frac{6}{8}$ of the area of the rectangle represented by the piece of paper.

Equivalent Fractions

Pictorially, one way the paper folding could look is in Figure 2.15.

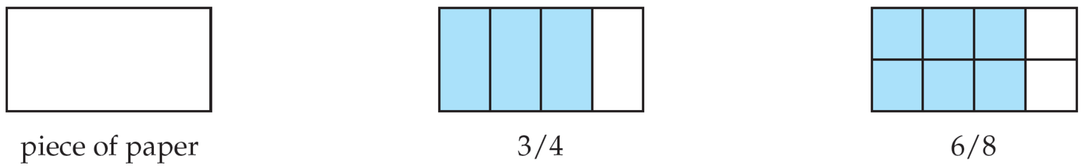
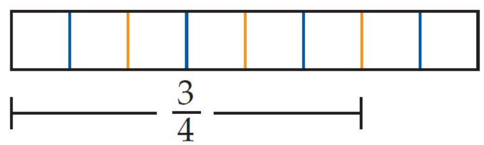


Figure 2.15

We can also use a linear model, like the bar model below. Here we take a linear bar, divide it into fourths (shown in orange), and then subdivide each fourth (shown in blue) to show that $\frac{3}{4} = \frac{6}{8}$.



Patterns in Equivalent Fractions

Patterns in Equivalent Fractions

Patterns in a sequence

Look at the sequence of equivalent fractions below. What patterns do you notice? What is the next fraction in the sequence? Why?

$$\frac{3}{4} = \frac{6}{8} = \frac{9}{12} = \frac{12}{16} \ldots$$

Some of the patterns include:

1. As we move from one fraction to another, the numerator increases by three and the denominator increases by four. In this sense, the $\frac{3}{4}$ acts as an operator.

Patterns in Equivalent Fractions

2. The numerator of each of the fractions is a multiple of three, and the denominator of each of the fractions is the same multiple of four.

3. All the denominators are even numbers, and every other numerator is an even number.

We can translate the first statement into notation:

Specific example: $\frac{3}{4} = \frac{3+3}{4+4} = \frac{6}{8}$

Patterns in Equivalent Fractions

More general case: $$\frac{a}{b} = \frac{\overbrace{a + a + \cdots + a}^{n\text{ times}}}{\underbrace{b + b + \cdots + b}_{n\text{ times}}} = \frac{an}{bn}$$

We can also translate the second statement into notation:

Specific example: $$\frac{3}{4} = \frac{3 \cdot 2}{4 \cdot 2} = \frac{6}{8}$$

More general case: $$\frac{a}{b} = \frac{a \cdot n}{b \cdot n} = \frac{an}{bn}$$

Addressing a Common Difficulty in Language

Addressing a Common Difficulty in Language

When trying to explain why $\frac{3}{4}$ and $\frac{6}{8}$ are equivalent, many students use the word *divide*—for example, “We divided the rectangle in half, and now there are 8 equal regions compared to 4 before.”

This choice of words is interesting and points to a problem that many children have in trying to understand equivalent fractions at a conceptual level. When we look at why fractions are equivalent, we commonly encounter this word *divide*.

Addressing a Common Difficulty in Language

However, in the procedure for creating equivalent fractions, we *multiply* the top and bottom by the same number! If we physically divide, why do we mathematically multiply?

The answer to this question has to do with the reciprocal relationship between multiplication and division.

Figure 2.16(a) shows $\frac{3}{4}$.

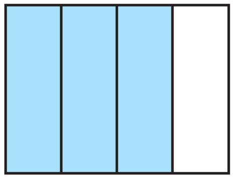


Figure 2.16(a)

Addressing a Common Difficulty in Language

If we divide each of the regions by 2, we are also multiplying the total number of regions by 2 [Figure 2.16(b)].

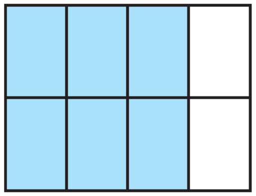


Figure 2.16(b)

We now have 6 out of 8 regions shaded—that is, $\frac{6}{8}$. Starting with $\frac{3}{4}$, we could have divided each region into 3 smaller pieces.

Addressing a Common Difficulty in Language

By dividing each region into 3 smaller pieces, we are multiplying the number of pieces by 3, and we can name this shaded amount $\frac{9}{12}$ [Figure 2.16(c)].

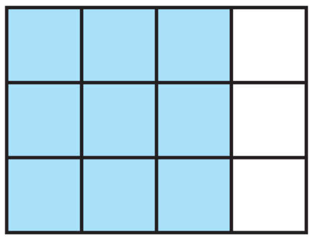


Figure 2.16(c)

Simplest Form

Simplest Form

A fraction is in **simplest form** if the numerator and the denominator have no common factors (other than 1). You may be used to the phrase of “reducing fractions.” The word “reduce” means to make smaller.

We are not making the value of the fraction smaller, we are just writing the fraction in a simpler way with smaller numbers. When children call this “reducing,” it can lead to the misconception that the quantity is being made smaller.

The notion of simplifying fractions is clearly connected to the concept of equivalent fractions.

Simplest Form

One important connection is that when we are simplifying fractions, we are essentially finding an equivalent fraction in which the numerator and denominator are smaller numbers. For example,

$$\frac{15}{20} = \frac{15 \div 5}{20 \div 5} = \frac{3}{4} \quad \text{or} \quad \frac{15}{20} = \frac{3 \times 5}{4 \times 5} = \frac{3}{4}$$

As has been true for other procedures, there are many strategies for simplifying fractions. Before we discuss them, simplify the following fractions yourself: $\frac{24}{40}$, $\frac{42}{60}$, and $\frac{63}{105}$.

Simplest Form

Explain your method for each.

One strategy is to divide the numerator and denominator by any common factor, not necessarily the greatest common factor. For example, you might divide both numerator and denominator by four, which produces $\frac{6}{10}$.

A quick glance reveals that this can be simplified further to $\frac{3}{5}$.

$$\frac{24}{40} = \frac{24 \div 4}{40 \div 4} = \frac{6}{10} = \frac{6 \div 2}{10 \div 2} = \frac{3}{5}$$

Simplest Form

Another strategy is to determine the prime factorization of each number and then cross out the common factors.

$$24 = \cancel{2} \cdot \cancel{2} \cdot \cancel{2} \cdot 3$$
$$40 = \cancel{2} \cdot \cancel{2} \cdot \cancel{2} \cdot 5$$

Let us examine more closely what happens when we are able to simplify the fraction in one step, as we did earlier to simplify $\frac{15}{20}$.

Simplest Form

What is the relationship between the divisor and the original numerator and denominator in each of the three fractions given above ($\frac{24}{40}$, $\frac{42}{60}$, and $\frac{63}{105}$)?

$$\frac{24 \div 8}{40 \div 8} = \frac{3}{5} \qquad \frac{42 \div 6}{60 \div 6} = \frac{7}{10} \qquad \frac{63 \div 21}{105 \div 21} = \frac{3}{5}$$

In each case, to simplify the fraction in one step, we divide both the numerator and the denominator by their greatest common factor. Let's explore this concept of greatest common factor a little more before continuing with fractions.

The Greatest Common Factor

The Greatest Common Factor

Think about what each of the words mean in this phrase "Greatest Common Factor." "Greatest," of course, means biggest. "Common" means what the two numbers have in common. The word "factor" means numbers that we can multiply to get a number.

As an example, the factors of the number 12 are {1, 2, 3, 4, 6,12} because

$$1 \times 12 = 12$$

$$2 \times 6 = 12$$

$$3 \times 4 = 12$$

The Greatest Common Factor

Factors of a number are whole numbers that we can multiply to get the number. The greatest of these common factors is called the **greatest common factor** (GCF).

We use the notation **GCF(*a*, *b*)** to express the GCF of two natural numbers *a* and *b*.

Investigation 2.2j – *Methods for Finding the Greatest Common Factor*

Let us now investigate how we might determine the Greatest Common Factor of two numbers.

A. Concrete/Pictorially find the Greatest Common Factor of 12 and 8.

Investigation 2.2j(a) – *Discussion*

Cuisenaire rods are a manipulative that help us see many concepts, including Greatest Common Factor. You can also find a virtual version of these at https://www.mathplayground.com/mathbars.html. How do these models help you see GCF of 12 and 8?

Using Cuisenaire rods, how many different ways can you make a 12 train and an 8 train using the same colors (Figure 2.17).

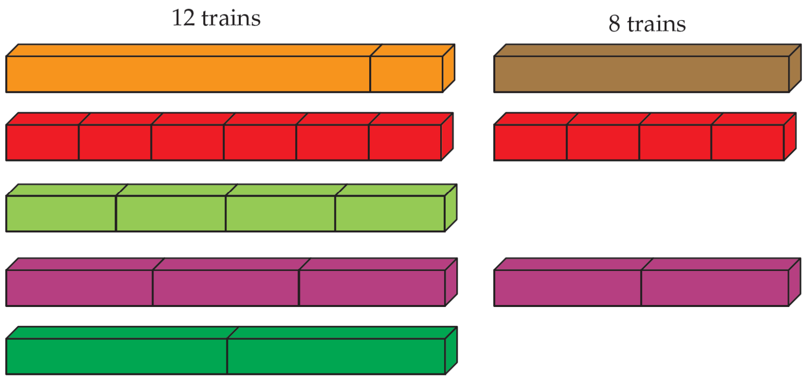


Figure 2.17

Investigation 2.2j(a) – *Discussion* continued

The red rods are 2 in length and the purple rods are 4 in length. Because we can use the red 2 rods and the purple 4 rods to make both the 8 and 12, this tells us that 2 and 4 are common factors of 8 and 12.

Because the purple 4 rods are the biggest we can use to build both 12 and 8, then 4 is the greatest common factor.

B. Using only the definition of Greatest Common Factor, how would you determine GCF (45, 60)? How does this relate to simplifying the fraction $\frac{45}{60}$? Work on this any way you want.

Investigation 2.2j(b) – *Discussion*

Strategy 1: Use factorization

We could, as we just saw, determine all the factors of each number and then find the largest of the common factors:

Factors of 45 = {1, 3, 5, 9,15, 45}

Factors of 60 = {1, 2, 3, 4, 5, 6,10,12,15, 20,30,60}

Common factors = {1, 3, 5,15}

We see from this list that 15 is the GCF of 45 and 60.

Investigation 2.2j(b) – *Discussion* continued

To simplify $\frac{45}{60}$, we could divide both the 45 and 60 by any of these common factors. However, dividing first by the greatest common factor takes the fraction to its simplest form in one step.

Strategy 2: Use intuition or number sense

A student who is highly intuitive and has good number sense might just know that 15 divides both these numbers. The fact that 15 divides both numbers simply means that 15 is a common factor. How might you reason that, in fact, 15 is the GCF? Think before reading on….

Investigation 2.2j(b) – *Discussion* continued

Let us represent the results of dividing each number by 5 (which we know is not the GCF) and by 15. What do you notice?

$45 = 5 \cdot 9$	$45 = 15 \cdot 3$
$60 = 5 \cdot 12$	$60 = 15 \cdot 4$

When we divide 45 and 60 by 5, we are left with 9 and 12. When we divide 45 and 60 by 15, we are left with 3 and 4. One difference between 9 and 12 and 3 and 4 is that 3 and 4 have no common factors other than 1.

Investigation 2.2j(b) – *Discussion* continued

When two numbers have no factors in common other than 1, they are said to be *relatively prime*. Because 3 and 4 are relatively prime, 15 is the GCF of 45 and 60. Do you see why?

Strategy 3: Repeatedly divide by prime numbers

Another procedure involves an adaptation of the long-division algorithm. The following problem illustrates a systematic application in that we begin with the smallest prime divisor and then move up. That is, we first divide both numbers by 3.

Investigation 2.2j(b) – *Discussion* continued

At this point, we move up to 5 because 15 and 20 are both divisible by 5. The resulting quotients, 3 and 4, have no factors in common. The GCF of 45 and 60 is the product of their common factors: 3 · 5 = 15.

$$\begin{array}{r} 3,\ \ 4\\ 5\overline{)15,\,20}\\ 3\overline{)45,\,60} \end{array}$$

Strategy 4: Use prime factorization

This strategy uses a method of breaking a number into its prime factors, called the **prime factorization**.

Investigation 2.2j(b) – *Discussion* continued

We first determine the prime factorization of each number and then look for common factors. If we look at the prime factorizations of 45 and 60 and circle the factors that the two numbers have in common, we have the following:

$$45 = 3 \cdot 3 \cdot 5$$
$$60 = 2 \cdot 2 \cdot 3 \cdot 5$$

We can further refine this procedure by using exponents:

$$45 = 3^2 \cdot 5^1$$
$$60 = 2^2 \cdot 3^1 \cdot 5^1$$

Investigation 2.2j(b) – *Discussion* continued

The GCF is determined by examining those factors that both numbers have in common and then taking the *smallest* exponent in each case. The common factors of 45 and 60 are 3 and 5. The smallest exponent of 3 is 1, and the smallest exponent of 5 is 1. Thus, $3 \cdot 5$ is the GCF.

Now, we have several ways to find the greatest common factor, which can help us to simplify fractions in one step. When working with a fraction like $\frac{45}{60}$, knowing the greatest common factor helps us to simplify this in one step by dividing by 15 and getting $\frac{3}{4}$.

Investigation 2.2j(b) – *Discussion* continued

When simplifying fractions, you can start with any common factor. If you do not start with the greatest common factor, you will just need to simplify again. This may be easier if both numbers making up the fraction are large. In the example of $\frac{45}{60}$, we could initially divide top and bottom by 5, which would give us $\frac{9}{12}$.

Then we would need to divide again by 3, since 9 and 12 have a common factor of 3. $\frac{9}{12} = \frac{3}{4}$. Notice that the greatest common factor of 45 and 60 is 15.

Investigation 2.2j(b) – *Discussion* continued

In the second method, we simplified by 5 and then 3; and it is not a coincidence that $3 \times 5 = 15$. In the first method we are dividing by 15 in one step, and in the second we are dividing by 15 in two steps, since $3 \times 5 = 15$.

Investigation 2.2k – *Sharing Cookies*

A. Aja came home after school one day and found that her mother had left a plate of cookies. Aja ate $\frac{1}{2}$ of the cookies. When her sister Nolise came home, she ate $\frac{1}{4}$ of the remaining cookies.

When their mother came home, there were 3 cookies on the plate. How many did each girl eat?

Investigation 2.2k – *Sharing Cookies* continued

B. Aja came home after school one day and found that her mother had left a plate of cookies. Aja ate $\frac{1}{2}$ of the cookies. When her sister Nolise came home, she ate $\frac{1}{3}$ of the remaining cookies, and when Clarise came home, she ate $\frac{1}{4}$ of the cookies on the plate. When their mother came home, there were 6 cookies on the plate. How many did each child eat?

Investigation 2.2k – *Discussion*

A. The diagram at the left represents the problem and each step. Aja ate $\frac{1}{2}$ of the cookies, and Nolise ate $\frac{1}{2}$ of what was left. If what remains is 3 cookies, that means that 3 cookies represents $\frac{1}{4}$ of what was originally there. Thus, there were 12 cookies; Aja ate 6, and Nolise ate 3.

B. Although we could use a similar area model as we used in part A, we will use a bar model to solve this one. First, we choose a bar to represent the entire cookie jar.

Investigation 2.2k – *Discussion* continued

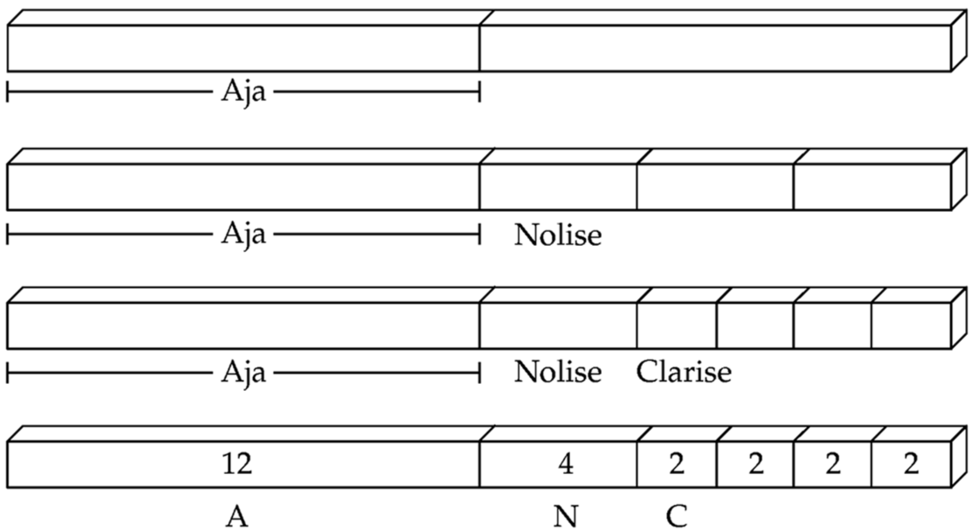
Then we mark off the $\frac{1}{2}$ that Aja ate. Then we mark off $\frac{1}{3}$ of the remaining to show what Nolise ate. Finally, we mark off the $\frac{1}{4}$ of the remaining that Clarise ate and put the 6 cookies evenly into the remaining three pieces, and we can fill the model from there.

The coding here helps us to communicate the solution.

STEPS: 1 2 3 4 5

Investigation 2.2k – *Discussion* continued

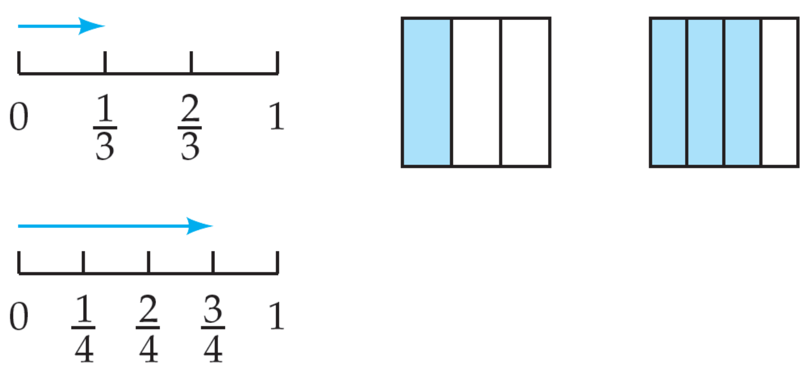
Aja ate $\frac{1}{2}$ of the cookies. Nolise ate $\frac{1}{3}$ of the remaining cookies, and Clarise ate $\frac{1}{4}$ of the remaining cookies. Then, there were 6 cookies remaining.



Equivalence, Benchmarks, and Fraction Sense

Equivalence, Benchmarks, and Fraction Sense

After children have explored fractions with various manipulatives, a common investigation is to give them pairs or groups of fractions and order them from least to greatest. Children begin by referring to physical models, $\frac{3}{4}$ is clearly greater than $\frac{1}{3}$, and this can be demonstrated with different models.



Equivalence, Benchmarks, and Fraction Sense

Visuals and manipulatives are powerful tools, but we also want students to develop more tools to navigate through problems and situations involving fractions. One such tool is equivalence. For example, can you explain why $\frac{2}{9} < \frac{1}{4}$ without having to make a picture?

Using equivalence, we can see that $\frac{1}{4} = \frac{2}{8}$. Now how do we know that $\frac{2}{9} < \frac{2}{8}$?

Equivalence, Benchmarks, and Fraction Sense

Using our knowledge of the meaning of fractions, we know that 9ths are smaller than 8ths, and since we have the same number of 9ths as 8ths, $\frac{2}{9}$ must be less than $\frac{2}{8}$.

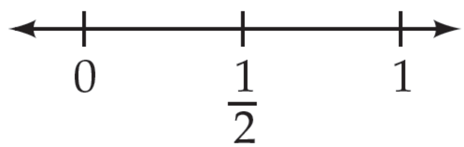
Children will often use the analogy of pizzas in this case: If you are hungry, you'd rather have 2 slices of a pizza that was divided into 8ths than 9ths.

We can use this idea of equivalences to establish benchmarks. Just as powers of 10 serve as benchmarks for whole numbers (1, 10, 100, 1000, 10,000, and so on), unit fractions, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and so on, serve as **benchmarks** to help us keep a sense of the size of fractions.

Equivalence, Benchmarks, and Fraction Sense

For example, which of these two fractions has a greater value: $\frac{3}{5}$ or $\frac{5}{12}$?

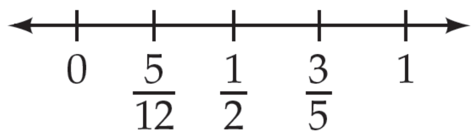
Where on this number line would $\frac{3}{5}$ and $\frac{5}{12}$ be?



Using $\frac{1}{2}$ as our benchmark helps us to determine this.

Equivalence, Benchmarks, and Fraction Sense

$\frac{3}{5}$ is greater than $\frac{1}{2}$ because $\frac{2\frac{1}{2}}{5} = \frac{1}{2}$ and $\frac{5}{12} < \frac{1}{2}$ because $\frac{6}{12} = \frac{1}{2}$, which puts $\frac{3}{5}$ to the right of $\frac{1}{2}$ and $\frac{5}{12}$ to the left of $\frac{1}{2}$ on the number line; therefore, $\frac{5}{12} < \frac{3}{5}$.



Investigation 2.2I – *Ordering Fractions*

Arrange these fractions from smallest to largest by focusing on fraction concepts and reasoning tools.

$$\frac{3}{4} \qquad \frac{2}{5} \qquad \frac{5}{6}$$

Investigation 2.2l – *Discussion*

There are a variety of ways in which we can validly answer the question. We will explore several of these ways with the intention of refining or expanding your "fraction sense" toolbox.

Some people start by picking two fractions whose order they know. In this example, let us start that way. We know that $\frac{2}{5} < \frac{3}{4}$.

An area diagram would readily show this, or using $\frac{1}{2}$ as a benchmark is another way to see this.

Investigation 2.2I – *Discussion* continued

We know that $\frac{3}{4}$ is greater than $\frac{1}{2}$ because 3 is more than $\frac{1}{2}$ of 4; similarly, we know that $\frac{2}{5}$ is less than $\frac{1}{2}$ because 2 is less than $\frac{1}{2}$ of 5. Thus, we can conclude that $\frac{2}{5} < \frac{3}{4}$.

The next debate concerns $\frac{5}{6}$ and $\frac{3}{4}$. How would you explain which is larger?

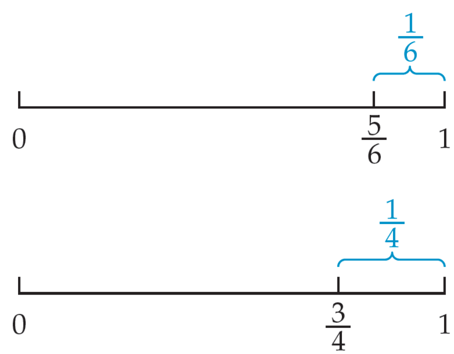


Figure 2.18

Investigation 2.2I – *Discussion* continued

Looking at the two fractions, we see that they are both one piece away from 1. Sixths are smaller than fourths, so we can reason that $\frac{5}{6}$ must be greater than $\frac{3}{4}$ because the distance between $\frac{5}{6}$ and 1 is $\frac{1}{6}$, whereas the distance between $\frac{3}{4}$ and 1 is $\frac{1}{4}$ (Figure 2.18). We can combine $\frac{2}{5} < \frac{3}{4}$ and $\frac{3}{4} < \frac{5}{6}$ to conclude that $\frac{2}{5} < \frac{3}{4} < \frac{5}{6}$.

Investigation 2.2m – *Estimating with Fractions*

In everyday life, we often see parts of a whole and are more interested in approximations than exact answers. For example, in 2012 the average salary for beginning public school teachers in the United States was $35,141, and the average salary for all public school teachers was $56,039.

We can look at these numbers from an additive perspective and say that the beginning teachers' salary is about $21,000 less than the average for all teachers. If we want to turn this into a fraction, we are now reasoning multiplicatively.

Investigation 2.2m – *Estimating with Fractions* continued

If we were to do this, we would ask, “Beginning teachers’ salaries is about what fraction of the overall.” Can you apply your understanding of fractions (and division!) to find a simple fraction (not $\frac{35}{56}$) that answers this question?

Investigation 2.2m – *Discussion*

To answer this question accurately, we need to apply three things we have learned:

Fraction can be seen as division.

When estimating division problems, we can round both numbers up or round both of them down. Look for compatible numbers.

$$\frac{35{,}141}{56039} \approx \frac{35{,}000}{56{,}000} = \frac{35}{56} = \frac{5}{8}$$

Mixed Numbers

Mixed Numbers

Up to this point, we have only looked at fractions whose value is between zero and one. In many reallife situations, we encounter mixed numbers (for example, $5\frac{1}{4}$) and improper fractions (for example, $\frac{21}{4}$). When we have a fraction that is bigger than one, we can express it as a mixed number as well.

A **mixed number** is a number that has a whole-number component and a fraction component.

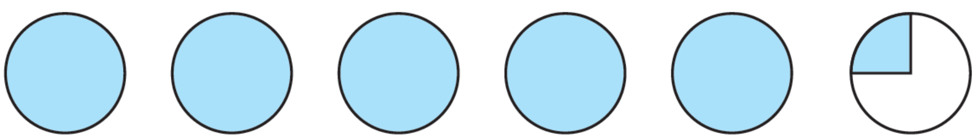
Mixed Numbers

A National Assessment of Educational Progress contained the following item:

$5\frac{1}{4}$ is the same as:

(a) $5+\frac{1}{4}$ (b) $5-\frac{1}{4}$ (c) $5\times\frac{1}{4}$ (d) $5-\frac{1}{4}$ (e) I don't know

We can draw a model of this to help by having five wholes and $\frac{1}{4}$ of another whole.



Mixed Numbers

Only 47 percent of the seventh-graders chose the correct response, (a). Still more startling, an even smaller percentage of eleventh-graders chose the correct response—only 44 percent.

This lack of connectedness between concepts and procedures can be helped with pictorial representations.

Investigation 2.2n – *Connecting Fractions and Mixed Numbers*

Try drawing an area model, a linear model, and a set model to explain why $3\frac{1}{4} = \frac{13}{4}$.

You may remember the procedure for converting a mixed number into an fraction. For example, to convert $3\frac{1}{4}$ into a fraction, we do the following:

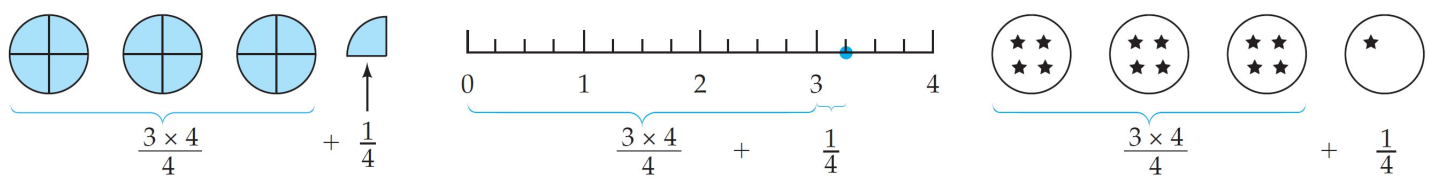
$$3\frac{1}{4} = \frac{3 \cdot 4 + 1}{4}$$

We multiply the *whole number* by the *denominator*, then we add the *numerator*, and then we put this number on top of the *original denominator*.

How would you explain the why of this procedure to, let's say, a fourth-grade student? How can the models you drew help explain this procedure?

Investigation 2.2n – *Discussion*

Examine the area, linear, and set models shown below. How are these similar and different from the ones you drew?



In each of the representations, we see that the process is similar to the kinds of regrouping we did with whole numbers.

Investigation 2.2n – *Discussion* continued

In converting from the mixed number to the fraction, we need to convert all the units to fourths because in order for the fraction, $\frac{13}{4}$, to have meaning, all the pieces must be the same size—that is, they must be fourths.

We can convert each one to 4 fourths and then add all the fourths to get $\frac{13}{4}$.

$$
\begin{aligned}
3\frac{1}{4} &= 1+1+1+\frac{1}{4} \\
&= \frac{4}{4}+\frac{4}{4}+\frac{4}{4}+\frac{1}{4} \\
&= \frac{3\cdot 4}{4}+\frac{1}{4} \\
&= \frac{3\cdot 4+1}{4} \\
&= \frac{13}{4}
\end{aligned}
$$

Investigation 2.2n – *Discussion* continued

However, because one context for multiplication is repeated addition, we see that we are adding three 4's; hence, three units times 4 fourths in each unit.

When we add these 12 fourths to the 1 fourth we already had, we have a total of 13 fourths. Notice in the set model that we chose to put four stars in each set since we were working with fourths.

The Density of the Set of Fractions

The Density of the Set of Fractions

This activity brings up a question that children sometimes ask: Can we name any point on the number line with a fraction? What do you think?.

For example, name a fraction between zero and one. If we did this with a whole class, we would get a number of correct responses, although $\frac{1}{2}$ might be the most common.

Can you name another fraction between zero and $\frac{1}{2}$? How many can you name?

The Density of the Set of Fractions

In fact, we can say that between any two fractions, there are an infinite number of fractions. Mathematicians refer to this property by saying that **fractions are dense**.

Think of naming any point on a number line, knowing that no matter how close two fractions are, we can find an infinite number of fractions between those two fractions!